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Computational Section

On a conjecture of Chen and Yui: Fricke groups ☆,☆☆

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In this work, we establish formulas for the prime decompositions of resultant and discriminant of the ring class polynomial associated with Thompson series for Fricke group $\Gamma_0(p)^+$ for p prime and imaginary quadratic field, and as a consequence, validate a conjecture of Chen and Yui on the upper bound of prime factors of such resultants and discriminants. All associated numerical conjectures of Chen and Yui are verified using the Magma code for our formulas written by Chao Qin.

☆ With an appendix by Chao Qin.

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1. Introduction

By the theory of complex multiplication of elliptic curves and class field theory, it is now widely known that the value of the modular j-invariant denoted by $j(z)$ for $z \in \mathbb{H}$ the upper half plane at the point $N\frac{-d+\sqrt{-d}}{2}$ for $-d$ a negative fundamental discriminant and N a positive integer generates the ring class field of conductor N over $\mathbb{Q}(\sqrt{-d})$, an object of great concern in algebraic number theory, and its Galois conjugates over $\mathbb{Q}$ are exactly $j(\tau_Q)$, where τ_Q is the unique imaginary quadratic point lying in $\mathbb{H}$ defined by a positive definite binary quadratic form $Q = Q(X, Y)$ via $Q(\tau, 1) = 0$, as $[Q]$ ranges over the representatives of $\mathcal{Q}_{N^2d}/\mathrm{SL}_2(\mathbb{Z})$, where $\mathcal{Q}_{N^2d}$ denotes the set of primitive positive definite integral binary quadratic forms of discriminant $-N^2d$, i.e.,

$$\mathcal{Q}_{N^2d} = \left\{aX^2 + bXY + cY^2 \,|\, a, b, c \in \mathbb{Z},\ a > 0,\ b^2 - 4ac = -N^2d\right\}.$$

These facts provide ones with a numerically feasible way to compute a defining polynomial $H_{d,N}(x)$ of the ring class field of conductor N over $\mathbb{Q}(\sqrt{-d})$, that is,

$$H_{d,N}(x) = \prod_{[Q]\in\mathcal{Q}_{N^2d}/\mathrm{SL}_2(\mathbb{Z})} (x - j(\tau_Q)),$$

called the ring class polynomial of conductor N associated with $j(z)$ and $\mathbb{Q}(\sqrt{-d})$, and such an interesting application sequentially and considerably brings the study of properties of such a particular family of ring class polynomials to ones' attention.

Implementing the aforementioned application, Berwick [1], and Ford and McKay [14, p. 349] respectively computed certain resultants and discriminants associated with $H_{d,1}(x)$ whose splitting field over $\mathbb{Q}(\sqrt{-d})$ is the so-called Hilbert class field. Their computations show that these rational integers are all incredibly highly factorizable, and their prime factors are bounded by a relatively very small quantity depending on d. These lead ones to suspect that the prime decompositions of resultant and discriminant associated with $H_{d,1}(x)$ may possess certain describable patterns. Shortly after Ford and McKay's discoveries, these patterns were first formulated by Gross and Zagier [11], who remarkably established explicit formulas for the prime decomposition of the resultant of $H_{d_1,1}(x)$ and $H_{d_2,1}(x)$ with d_1, d_2 coprime, and that of the discriminant of $H_{d,1}(x)$ with d a prime congruent to 3 modulo 4. As an implication, these formulas follow that the prime factors of the these rational integers are respectively no greater than $\frac{d_1d_2}{4}$ and d, and thus, explain Berwick et al.'s observations.

Gross and Zagier's seminal work later inspired a number of sequential research on the topic. In [7], generalizing Gross and Zagier's algebraic proof of their formula for resultant, Dorman obtains an equivalent algebraic formulation, and in recent work [17], Lauter and Viray extend it to the case for arbitrary ring class polynomials $H_{d_1,N_1}(x)$ and $H_{d_2,N_2}(x)$ with $N_1^2d_1 \neq N_2^2d_2$. Also, Dorman [8] extended Gross and Zagier's formula for the discriminant of $H_{d,1}(x)$ to the case that $-d$ is an arbitrary negative fundamental discriminant, and Hayashi further extended it to the case for arbitrary ring class polynomial

$H_{d,N}(x)$ in his thesis [12]. Recently, using a fundamentally different approach, we [24] also establish an equivalent formulation for the prime decomposition of the discriminant of $H_{d,N}(x)$, and consequently, re-prove Dorman's formula.

On the other hand, as a member of the family of Thompson series, a family of uniformizers of modular curves for Fuchsian groups associated with the monstrous moonshine, it is commonly believed that many properties of the modular j-invariant $j(z)$ also hold for other Thompson series. For instance, similar to the algebraic property of $j(z)$ stated at the beginning, it was proved [6] by Chen and Yui that the value of a Thompson series $T(\tau)$ of level N at the point $\frac{-d+\sqrt{-d}}{2}$ for $-d$ a negative fundamental discriminant generates the ring class field of conductor N over $\mathbb{Q}(\sqrt{-d})$ with Galois conjugates exactly $T(\tau_Q)$ as $[Q]$ ranges over $\mathcal{Q}_d(N)/\Gamma_0(N)$, where $\mathcal{Q}_d(N)$ denotes the set of primitive positive definite binary quadratic form $aX^2+bXY+cY^2$ of discriminant $-d$ with $\gcd(a,N)=1$, i.e.,

$$\mathcal{Q}_d(N)=\left\{aX^2+bXY+cY^2 \,|\, a,b,c\in\mathbb{Z},\ a>0,\ b^2-4ac=-d,\ \gcd(a,N)=1\right\},$$

and this leads to another feasible way of computing ring class polynomials of conductor N associated with $\mathbb{Q}(\sqrt{-d})$ over $\mathbb{Q}$. Inspired by Berwick et al.'s work, like one may naturally look into the resultants and discriminants of these polynomials, Chen and Yui did numerically compute a large number of examples, and found that similar to what we have just seen on $j(z)$, the numbers are also all highly factorizable. For example, letting $H^+_{d,N}(x)$ denote the ring class polynomial of conductor N associated with the Thompson series $j^+_N(z)$ for Fricke group $\Gamma_0(N)^+=\left\langle \Gamma_0(N), w_N := \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}\right\rangle$ and $\mathbb{Q}(\sqrt{-d})$, i.e.,

$$H^+_{d,N}(x)=\prod_{[Q]\in\mathcal{Q}_d(N)/\Gamma_0(N)}\left(x-j^+_N(\tau_Q)\right), \tag{1.1}$$

their computations showed that

$$H^+_{3,7}(x)=x^2+224x+448 \quad\text{and}\quad H^+_{4,7}(x)=x^4-528x^3-9024x^2-5120x-1728,$$

where

$$j^+_7(\tau)=\frac{1}{q}\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1-q^{7n}}\right)^4+7^2q\prod_{n=1}^{\infty}\left(\frac{1-q^{7n}}{1-q^n}\right)^4 \quad \text{with } q=\exp(2\pi i\tau),$$

and consequently, the resultant

$$\left|\text{result}\left(H^+_{3,7}(x),H^+_{4,7}(x)\right)\right|=2^{12}3^6 11^2 47^1 83^1 131^1, \tag{1.2}$$

and the discriminant

$$\left|\operatorname{disc}\left(H_{3,7}^{+}(x)\right)\right| = 2^8 3^3 7^1. \tag{1.3}$$

Based on careful inspection on their abundant numerical data, Chen and Yui predicted that prime factorization formulas analogous to Gross and Zagier's may also exist for Thompson series of level N, and accordingly conjectured that the prime factors involved are bounded by a certain quantity depending on d and N. For example, they made the following conjecture.

Conjecture 1.1. *Let p be a prime such that $\Gamma_0(p)^+$ is of genus zero. Let $\operatorname{result}\Big(H_{d_1,p}^{+}(x), H_{d_2,p}^{+}(x)\Big)$ denote the resultant of the ring class polynomials $H_{d_1,p}^{+}(x)$ and $H_{d_2,p}^{+}(x)$, and let $\operatorname{disc}\Big(H_{d,p}^{+}(x)\Big)$ denote the discriminant of the ring class polynomial $H_{d,p}^{+}(x)$. Then*

(1) the prime factors of $\operatorname{result}\Big(H_{d_1,p}^{+}(x), H_{d_2,p}^{+}(x)\Big)$ with $(d_1, d_2) = 1$ are bounded by $\frac{p^2 d_1 d_2}{4}$;
(2) the prime factors of $\operatorname{disc}\Big(H_{d,p}^{+}(x)\Big)$ are bounded by dp.

Remark 1.2. By the Riemann–Hurwitz formula, there are only finitely many primes p for which $\Gamma_0(p)^+$ are of genus zero, and these are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71,$$

whose Thompson series $j_p^{+}(\tau)$ are correspondingly uniformizers of the modular curves of $\Gamma_0(p)^+$ with a simple pole at the cusp $[i\infty]$ and can be explicitly found in [5, Tables 3 and 4a] or [13, Table 3].

In recent work [21], we realize Chen and Yui's prediction on Thompson series for $\Gamma_0(p)$ for p prime. In the present work, we treat the case for Thompson series for $\Gamma_0(p)^+$, and ultimately validate Conjecture 1.1. To this end, we shall establish explicit prime factorization formulas for resultants and discriminants. The former case is now summarized in the following theorem.

Theorem 1.3. *Let p be a prime such that $\Gamma_0(p)^+$ is of genus zero. Let $-d_1, -d_2$ be two coprime negative fundamental discriminants and let $\chi_{d_i}(\cdot) = \left(\frac{-d_i}{\cdot}\right)$ be the associated quadratic character. Let $H_{d,p}^{+}(x)$ be the ring class polynomial of conductor p be defined as in (1.1). Then one has that*

$$\log\left|\operatorname{result}\Big(H_{d_1,p}^{+}(x), H_{d_2,p}^{+}(x)\Big)\right| = \sum_{\ell \text{ prime}} \mathfrak{e}_\ell \log \ell,$$

where

$$\mathfrak{e}_\ell = \frac{(p-2-2\chi_{d_1}(p)-\chi_{d_2}(p))}{2}\left(\sum_{x^2<d_1d_2} F_{1,\ell}\left(\frac{d_1d_2-x^2}{4}\right)\right)$$
$$+ C(p,d_1,d_2)\left(\sum_{x^2<d_1d_2} F_{1,\ell}\left(\frac{d_1d_2-x^2}{4p}\right)\right)$$
$$+\frac{1}{2}\sum_{x^2<p^2d_1d_2}\begin{cases} F_{p,\ell}\left(\frac{p^2d_1d_2-x^2}{4}\right) & \text{if } \ell \neq p,\\ \mathfrak{A}_{p,p,2}\left(\frac{(p^2d_1d_2-x^2)}{4}\right) & \text{if } \ell = p,\end{cases}$$

and

$$C(p,d_1,d_2) = \frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(2-\chi_{d_1}(p))(2-\chi_{d_2}(p))}{4}.$$

Here $\operatorname{sgn}(r)$ *denotes the sign function defined to be 1 if* $r \geq 0$*, and* -1 *otherwise,*

$$F_{l,\ell}(m) = \frac{1}{e_\ell}\sum_{r\geq 1}\mathfrak{A}_{l,\ell,r}(m),$$

e_ℓ *denotes the ramification degree of* ℓ *in* $\mathbb{Q}(\sqrt{-d_1})$*, and*

$$\mathfrak{A}_{l,\ell,r}(m) = \epsilon_\ell\left(\frac{m}{\ell^r}\right)\prod_{\substack{q \text{ prime}\\ q|m,\, q\neq\ell}}\begin{cases} 1+\operatorname{ord}_q(m) & \text{if } \chi_{d_1}(q)=1,\ q\neq l,\\ 2 & \text{if } \chi_{d_1}(q)=1,\ q=l,\\ \frac{1}{2}\left(1+(-1)^{\operatorname{ord}_q(m)}\right) & \text{if } \chi_{d_1}(q)=-1,\ q\neq l,\\ 2 & \text{if } q|d_1,\ (-d_1,-m)_q=1,\ q\neq l,\\ 1 & \text{if } q|d_1,\ (-d_1,-m)_q=1,\ q=l,\ \operatorname{ord}_q(m)\leq 2,\\ 0 & \text{otherwise,}\end{cases}$$

where

$$\epsilon_\ell(N) = \begin{cases} 0 & \text{if either } N\notin\mathbb{Z}, \text{ or } \ell\nmid pd_1 \text{ and } \operatorname{ord}_\ell(N)\equiv 1 \pmod 2,\\ 2 & \text{if } N\in\mathbb{Z} \text{ and } \ell|d_1,\\ 1 & \text{otherwise,}\end{cases}$$

and $(-d_1,-m)_q$ *denotes the local Hilbert symbol at the place* q*.*

Remark 1.4. We remark that in the definition of $\mathfrak{A}_{l,\ell,r}(m)$, the product is defined to be over all prime factors of m not equal to ℓ. So when $l = \ell$, the second case is by default automatically omitted.

Remark 1.5. It is noteworthy that what we compute in the present work are different from those in [9] by Gross, Kohnen and Zagier, in which they indeed compute and derive formulas for the prime factorizations of the resultants of class polynomials $\mathcal{H}^+_{d,p}(x)$ associated with $j^+_p(\tau)$ and Heegner points, i.e.,

$$\mathcal{H}^+_{d,p}(x) = \prod_{[Q]\in\mathcal{Q}_{d,p}/\Gamma_0(p)} \left(x - j^+_p(\tau_Q)\right),$$

where

$$\mathcal{Q}_{d,p} = \{aX^2 + bXY + cY^2 |\, aX^2 + bXY + cY^2 \in \mathcal{Q}_d,\ p|a\}.$$

Example 1.6. Take $p = 7$, $-d_1 = -3$ and $-d_2 = -4$. By Theorem 1.3 and the definition of $\mathfrak{e}_\ell$, one can see that $\mathfrak{e}_\ell \neq 0$ only if ℓ divides $\frac{d_1d_2-x^2}{4}$, $\frac{d_1d_2-x^2}{4p}$ or $\frac{p^2d_1d_2-x^2}{4}$ for some x such that either one is positive integral, or $\ell = p$, and thus by the choices of p, $-d_1$ and $-d_2$, one first has that

$$\begin{aligned}&\log\left|\text{result}\left(H^+_{3,7}(x), H^+_{4,7}(x)\right)\right|\\ &= \mathfrak{e}_2\log 2 + \mathfrak{e}_3\log 3 + \mathfrak{e}_7\log 7 + \mathfrak{e}_{11}\log 11 + \mathfrak{e}_{13}\log 13 + \mathfrak{e}_{23}\log 23 + \mathfrak{e}_{37}\log 37\\ &\quad + \mathfrak{e}_{47}\log 47 + \mathfrak{e}_{61}\log 61 + \mathfrak{e}_{73}\log 73 + \mathfrak{e}_{83}\log 83 + \mathfrak{e}_{131}\log 131.\end{aligned}$$

Using the formula for $\mathfrak{e}_\ell$, one can compute and obtain that

$$\begin{gathered}\mathfrak{e}_2 = 12, \quad \mathfrak{e}_3 = 6, \quad \mathfrak{e}_7 = 0, \quad \mathfrak{e}_{11} = 2, \quad \mathfrak{e}_{13} = 0, \quad \mathfrak{e}_{23} = 0,\\ \mathfrak{e}_{37} = 0, \quad \mathfrak{e}_{47} = 1, \quad \mathfrak{e}_{61} = 0, \quad \mathfrak{e}_{73} = 0, \quad \mathfrak{e}_{83} = 1, \quad \mathfrak{e}_{131} = 1,\end{gathered}$$

and thus, obtains

$$\left|\text{result}\left(H^+_{3,7}(x), H^+_{4,7}(x)\right)\right| = 2^{12}3^6 11^2 47^1 83^1 131^1,$$

which coincides with (1.2).

Remark 1.7. Using the Magma code for Theorem 1.3 listed in Appendix, we have verified that for the cases considered on page 316 of [6] by Chen and Yui, the prime factorizations yielded by Theorem 1.3 all correspondingly match with those given in the table therein. Note that there is a typo in their table on the case $(p, d_1, d_2) = (7, 7, 8)$ for which the exponent of the prime factor 13 should be 14 instead of 4, and this can be checked by directly computing the resultant of the class polynomials

$$H^+_{7,7}(x) = x^7 + 4046x^6 - 64799x^5 + 16442335x^4 + 14883071x^3 + 199370017x^2 - 45950625x + 3^6 5^6$$

and

$$H^+_{8,7}(x) = x^8 - 7328x^7 + 655872x^6 + 1089536x^5 + 155062656x^4 - 53286912x^3 - 60612608x^2 + 81920000x + 2^{12}5^6$$

given in the table on Page 300 of [6].

The following corollary of Theorem 1.3 affirms Conjecture 1.1 (1).

Corollary 1.8. *Any prime factor of the resultant of* $H^+_{d_1,p}(x)$ *and* $H^+_{d_2,p}(x)$ *with* p *prime and* $\gcd(d_1, d_2) = 1$ *is less than* $\frac{p^2 d_1 d_2}{4}$.

In what follows, we enunciate the formula for the prime factorization of the discriminant of $H^+_{q,p}(x)$ with $q \equiv 3 \pmod 4$ a prime not equal to p, i.e., $-q$ a fundamental discriminant. See Theorem 3.11 for the general case.

Theorem 1.9. *Let* $p \geq 3$ *be a prime such that* $\Gamma_0(p)^+$ *is of genus zero, and let* $q \equiv 3 \pmod 4$ *be a prime not equal to* p. *Then*

$$\log \left|\mathrm{disc}\left(H^+_{q,p}(x)\right)\right| = \sum_{\ell \text{ prime}} \mathfrak{e}_\ell \log \ell,$$

where

$$\mathfrak{e}_\ell = \sum_{\substack{[\mathfrak{a}] \in \mathrm{Cl}_q(p) \\ [\mathfrak{a}] \neq [\mathcal{O}_q]}} \mathfrak{e}_{\ell,\mathfrak{a}},$$

and

$$\mathfrak{e}_{\ell,\mathfrak{a}} = \sum_{l=0}^{pq-1} \sum_{X,Y=-\infty}^{\infty} G_{\ell,\mathfrak{a},pl}\left(\frac{4Aq - q(2AX + pBY)^2 - (qpY - 2Al)^2}{4Aq}\right)$$
$$+ \sum_{\substack{1 \leq j,k \leq p-1 \\ jk \equiv -1 \pmod p}} \sum_{l=0}^{pq-1}$$
$$\times \sum_{X,Y=-\infty}^{\infty} G_{\ell,\mathfrak{a},j}\left(\frac{4Aqp - qp^2(2AX + pBY)^2 - (qp^2Y - 2(pCk - j + pl)A)^2}{4Aqp^2}\right),$$

the ideal class representatives $\mathfrak{a}$ *are chosen to be* $\left[A, \frac{B+\sqrt{-q}}{2}\right]$ *so that* $\gcd(A, pq) = 1$*, and the function* $G_{\ell,\mathfrak{a},n}(m)$ *is defined as follows (all the products involved are over primes prescribed by the corresponding given assumption).*

(i) For $m < 0$*,* $G_{\ell,\mathfrak{a},n}(m) = 0$*. Note that this actually implies that* $\mathfrak{c}_{\ell,\mathfrak{a}}$*'s are all finite sums, and see also Remark 1.10.*

(ii) For $m = 0$ *and* n *such that* $p|n$ *and* $p^2 \nmid n$*,*

(a) if $\ell \neq p$*,* $G_{\ell,\mathfrak{a},n}(0) = 0$*,*

(b) otherwise, $G_{\ell,\mathfrak{a},n}(0) = \frac{h_q}{2}$*, where* h_q *denotes the class number of* $\mathbb{Q}(\sqrt{-q})$ *divided by half of the number of the integral units of the field.*

(iii) For $m > 0$ *and*

(a) for $\ell \neq p, q$*,*

$$\begin{aligned} &G_{\ell,\mathfrak{a},n}(m) \\ &= -\frac{1}{2}\prod_{t\nmid \ell pq}\left(\sum_{j=0}^{\mathrm{ord}_t(m)} \chi_q(t)^j\right) W_{\mathfrak{a}}(0,m,n)\left(\sum_{j=1}^{\mathrm{ord}_\ell(m)} \chi_q(\ell)^j j\right) \\ &\quad \times \begin{cases} 1 & \text{if } \mathrm{ord}_q(m) \leq -1, \\ \left(1 + \left(\frac{-m/q^{\mathrm{ord}_q(m)}}{q}\right)^{\mathrm{ord}_q(m)}\right) & \text{otherwise,} \end{cases} \end{aligned}$$

(b) for $\ell = q$*,*

$$\begin{aligned} G_{\ell,\mathfrak{a},n}(m) = &-\frac{1}{2}\prod_{t\nmid \ell pq}\left(\sum_{j=0}^{\mathrm{ord}_t(m)} \chi_q(t)^j\right) \\ &\times W_{\mathfrak{a}}(0,m,n)\left(\frac{-m/q^{\mathrm{ord}_q(m)}}{q}\right)^{\mathrm{ord}_q(m)} (\mathrm{ord}_q(m)+1), \end{aligned}$$

(c) for $\ell = p$*,*

$$\begin{aligned} G_{\ell,\mathfrak{a},n}(m) = &-\frac{1}{2}\prod_{t\nmid \ell pq}\left(\sum_{j=0}^{\mathrm{ord}_t(m)} \chi_q(t)^j\right) \frac{W'_{\mathfrak{a}}(0,m,n)}{\log p} \\ &\times \begin{cases} 1 & \text{if } \mathrm{ord}_q(m) \leq -1, \\ \left(1 + \left(\frac{-m/q^{\mathrm{ord}_q(m)}}{q}\right)^{\mathrm{ord}_q(m)}\right) & \text{otherwise,} \end{cases} \end{aligned}$$

where

(1) for $p^2|n$,

$$W_{\mathfrak{a}}(s,m,n) = \begin{cases} 1+\left(\frac{-Am}{p}\right)p^{-s} & \text{if } \mathrm{ord}_p(m)=0, \\ 1+(p-1)\left(\sum_{j=2}^{\mathrm{ord}_p(m)}\left(\chi_q(p)p^{-s}\right)^j\right)-\left(\chi_q(p)p^{-s}\right)^{\mathrm{ord}_p(m)+1} & \text{if } \mathrm{ord}_p(m)\geq 1, \\ 1+(p-1)\sum_{n=2}^{\infty}\chi_q(p)^n p^{-ns} & \text{otherwise}, \end{cases}$$

(2) for $p|n$ *and* $p^2 \nmid n$,

$$W_{\mathfrak{a}}(s,m,n) = 1+\left(\frac{-Aq\left(mq+A(n/p)^2\right)}{p}\right)p^{-s},$$

(3) for $p \nmid n$,

$$W_{\mathfrak{a}}(s,m,n) = 1,$$

Remark 1.10. Since $G_{\ell,\mathfrak{a},n}(m)$ is defined to be 0 for $m<0$, one can see that the innermost sums of $\mathfrak{e}_{\ell,\mathfrak{a}}$ are finite sums. More precisely, one can see that

$$G_{\ell,\mathfrak{a},pl}\left(\frac{4Aq-q(2AX+pBY)^2-(qpY-2Al)^2}{4Aq}\right)$$

and

$$G_{\ell,\mathfrak{a},j}\left(\frac{4Aqp-qp^2(2AX+pBY)^2-(qp^2Y-2(pCk-j+pl)A)^2}{4Aqp^2}\right)$$

contribute to $\mathfrak{e}_\ell$ only if

$$4Aq-q(2AX+pBY)^2-(qpY-2Al)^2 \geq 0$$

and

$$4Aqp-qp^2(2AX+pBY)^2-(qp^2Y-2(pCk-j+pl)A)^2 \geq 0,$$

respectively. These induce bounds on Y's as one correspondingly has that

$$(qpY-2Al)^2 \leq 4Aq-q(2AX+pBY)^2 \leq 4Aq$$

and

$$(qp^2Y - 2(pCk - j + pl)A)^2 \leq 4Aqp - qp^2(2AX + pBY)^2 \leq 4Aqp,$$

and subsequently, bounds on X's.

Remark 1.11. We refer the reader to Remark 3.8 for the reason why it is not necessary to include all cases for (m, n) in Theorem 1.9.

Example 1.12. Take $p = 7$ and $-d = -3$. One can check that $\mathrm{Cl}_d(p) = \{[\mathcal{O}_d], [\mathfrak{a}]\}$ with $\mathfrak{a} = \left[31, \frac{11+\sqrt{-3}}{2}\right]$. By the definition of $G_{\ell,\mathfrak{a},n}(m)$, one can tell that it does not vanish only if $\mathrm{ord}_\ell(m) > 0$, $\mathrm{ord}_\ell(md) > 0$, or $m = 0$, and thus, one obtains by Theorem 1.9 that

$$\log \left|\mathrm{disc}\left(H^+_{3,7}(x)\right)\right| = \mathfrak{e}_2 \log 2 + \mathfrak{e}_3 \log 3 + \mathfrak{e}_7 \log 7,$$

where

$$\begin{aligned}
\mathfrak{e}_2 &= \sum_{l=4,7,10,11,14,17} G_{2,\mathfrak{a},7l}\left(\frac{2}{3}\right) + 2\sum_{j=2,5} G_{2,\mathfrak{a},j}\left(\frac{2}{147}\right) + \sum_{j=1,6} G_{2,\mathfrak{a},j}\left(\frac{6}{49}\right) \\
&\quad + 2\sum_{j=3,4} G_{2,\mathfrak{a},j}\left(\frac{8}{147}\right), \\
\mathfrak{e}_3 &= G_{3,\mathfrak{a},0}(1) + \sum_{j=2,5} G_{3,\mathfrak{a},j}\left(\frac{3}{49}\right), \\
\mathfrak{e}_7 &= 2\sum_{l=3,18} G_{7,\mathfrak{a},7l}(0) + \sum_{l=6,15} G_{7,\mathfrak{a},7l}(0).
\end{aligned}$$

Following the definition of $G_{\ell,\mathfrak{a},n}(m)$, one can compute and show that

$$\begin{aligned}
\mathfrak{e}_2 &= 6 \times 0 + 2 \times 2 \times \frac{1}{2} + 2 \times 1 + 2 \times 2 \times 1 = 8, \\
\mathfrak{e}_3 &= 1 + 2 \times 1 = 3, \\
\mathfrak{e}_7 &= 2 \times 2 \times \frac{1}{6} + 2 \times \frac{1}{6} = 1,
\end{aligned}$$

and thus,

$$\left|\mathrm{disc}\left(H^+_{3,7}(x)\right)\right| = 2^8 3^3 7^1,$$

which coincides with (1.3).

Using Theorem 1.9, one can obtain Corollary 1.13 and partially confirm Conjecture 1.1 (2). See Corollary 3.12 for the general case.

Corollary 1.13. *Any prime factor of the discriminant of $H^+_{q,p}(x)$ with p, q distinct odd primes is less than pq.*

Before we proceed to the next section, we list some of the notation that shall be frequently adopted in the remainder of this work and their definitions as follows.

- $\mathcal{Q}_{d,p}$: the set of primitive positive definite integral binary quadratic forms of discriminant $-d$ with leading coefficient divisible by p, i.e.,

$$\{aX^2+bXY+cY^2 \,|\, aX^2+bXY+cY^2 \in \mathcal{Q}_d,\, p|a\},$$

- $\mathcal{Q}_{d,p,\beta}$: the subset of quadratic forms $aX^2+bXY+cY^2$ of $\mathcal{Q}_{d,p}$, for which $b \equiv \beta \pmod{2p}$ for some fixed $\beta \pmod{2p}$ such that $-d \equiv \beta^2 \pmod{4p}$, i.e.,

$$\{aX^2+bXY+cY^2 \,|\, aX^2+bXY+cY^2 \in \mathcal{Q}_{d,p},\, b \equiv \beta \pmod{2p}\},$$

- h_d: the class number of $\mathbb{Q}(\sqrt{-d})$ of fundamental discriminant $-d<0$ divided by half the size of its group of units, i.e.,

$$h_d = \begin{cases} \frac{1}{3} & \text{if } d=3, \\ \frac{1}{2} & \text{if } d=4, \\ \text{the class number of } \mathbb{Q}(\sqrt{-d}) & \text{if } d>4, \end{cases}$$

- $h_d(p)$: the form class number of $\mathcal{Q}_d(p)$ modulo the group action of $\Gamma_0(p)$, i.e., $h_d(p) = |\mathcal{Q}_d(p)/\Gamma_0(p)|$,
- Γ_Q: the stabilizer subgroup of a quadratic form Q of Γ, i.e., $\Gamma_Q = \{\gamma \in \Gamma \,|\, Q\cdot\gamma = Q\}$,
- $\zeta(s)$: the Riemann zeta function,
- $\zeta_d(s)$: the Dedekind zeta function associated with $\mathbb{Q}(\sqrt{-d})$ of fundamental discriminant $-d<0$,
- $\mathrm{Cl}_d(p)$: the ring class group of conductor p of $\mathbb{Q}(\sqrt{-d})$ of fundamental discriminant $-d<0$, i.e., the group generated by integral ideals of norm prime to p modulo its subgroup generated by principal integral ideals of norm prime p.

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2. Proof of Theorem 1.3

This section is devoted to proving Theorem 1.3, and throughout this section, $-d_1, -d_2$ are assumed to be coprime negative fundamental discriminants, and $-d_1$ is assumed to be odd, i.e., square-free, by symmetry. As the proof involves various concepts and results, to avoid an unwieldy long exposition, we break the section into several subsections.

2.1. Automorphic Green functions

By (1.1), it is clear that

$$\log\left|\text{result}\left(H^+_{d_1,p}(x), H^+_{d_2,p}(x)\right)\right|^2 = \sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)} \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \log|j^+_p(\tau_{Q_1}) - j^+_p(\tau_{Q_2})|^2,$$

where the corresponding harmonic function $\log\left|j^+_p(z_1) - j^+_p(z_2)\right|^2$ can be related to the so-called automorphic Green functions as follows, whose proofs can be found in [9, Proposition 1, p. 544] and [23].

Lemma 2.1. *Let $g_s(z_1, z_2)$ be the Green resolvent defined by*

$$g_s(z_1, z_2) = -2Q_{s-1}\left(1 + \frac{|z_1 - z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}\right),$$

where

$$Q_{s-1}(t) = \int_0^\infty \left(t + \sqrt{t^2-1}\cosh v\right)^{-s} dv$$

defined for $\mathrm{Re}(s) > 0$ *and* $t > 1$*. For $p = 1$ or a prime, write*

$$G_{p,s}(z_1, z_2) = \sum_{\gamma\in\Gamma_0(p)} g_s(z_1, \gamma\cdot z_2)$$

and let $E_p(z, s)$ be the non-holomorphic Eisenstein series of weight 0 for $\Gamma_0(p)$ defined by

$$E_p(z, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \mathrm{Im}(\gamma\cdot z)^s, \tag{2.1}$$

where $\Gamma_\infty = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -I \right\rangle$*. Write*

$$E^*_p(z, s) = E_p(z, s) + E_p(w_p\cdot z, s).$$

Define $\phi(s)$ to be

$$\phi(s) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

Then one has that for $p = 1$,

$$\log|j(z_1)-j(z_2)|^2 = \lim_{s\to 1}\left\{G_{1,s}(z_1,z_2)+4\pi\left(E_1(z_1,s)+E_2(z_2,s)-\phi(s)-\frac{6}{\pi}\right)\right\}, \tag{2.2}$$

and for p prime,

$$\begin{aligned}&\log|j_p^+(z_1)-j_p^+(z_2)|^2\\ &=\lim_{s\to 1}\Bigg\{G_{p,s}(z_1,z_2)+G_{p,s}(z_1,w_p\cdot z_2) \\ &\qquad -\frac{4\pi}{1-2s}\left(E_p^*(z_1,s)+E_p^*(z_2,s)-\left(p^{1-2s}\frac{1-p^{-1}}{1-p^{-2s}}+p^{-s}\frac{1-p^{-2s+1}}{1-p^{-2s}}\right)\phi(s)\right)\Bigg\}.\end{aligned}\tag{2.3}$$

Then as an instant implication of (2.3), one has that

$$\begin{aligned}&\log\left|\mathrm{result}\left(H^+_{d_1,p}(x),H^+_{d_2,p}(p)\right)\right|^2\\ &=\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}\log|j_p^+(\tau_{Q_1})-j_p^+(\tau_{Q_2})|^2\\ &=\lim_{s\to 1}\Bigg\{\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\ &\qquad +\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},w_p\cdot\tau_{Q_2})\\ &\qquad -\frac{4\pi}{1-2s}\Bigg(h_{d_2}(p)\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}E_p^*(\tau_{Q_1},s)+h_{d_1}(p)\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}E_p^*(\tau_{Q_2},s)\\ &\qquad\qquad -h_{d_1}(p)h_{d_2}(p)\left(p^{1-2s}\frac{1-p^{-1}}{1-p^{-2s}}+p^{-s}\frac{1-p^{-2s+1}}{1-p^{-2s}}\right)\phi(s)\Bigg)\Bigg\},\end{aligned}\tag{2.4}$$

and thus, to compute the resultant is equivalent to evaluating the limit on the right hand side of (2.4), which comprises three main components,

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},\tau_{Q_2}) \tag{2.5}$$

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},w_p\cdot\tau_{Q_2}) \tag{2.6}$$

$$\sum_{[Q]\in\mathcal{Q}_{d_i}(p)/\Gamma_0(p)}E_p^*(\tau_Q,s). \tag{2.7}$$

The treatment of the first two components (2.5) and (2.6) is closely related to the following formulas due to Gross, Kohnen and Zagier [9].

Lemma 2.2 *(Gross, Kohnen and Zagier). Let p be 1 or a prime such that $\Gamma_0(p)^+$ is of genus zero and let $-d_1, -d_2$ be two coprime negative fundamental discriminants. Then one has that*

$$\lim_{s\to 1}\left(\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})}\frac{4}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}||\mathrm{SL}_2(\mathbb{Z})_{Q_2}|}G_{1,s}(\tau_{Q_1},\tau_{Q_2})\right.$$
$$\left.+4\pi\left(h_{d_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)}+h_{d_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)}-h_{d_1}h_{d_2}\phi(s)-h_{d_1}h_{d_2}\frac{6}{\pi}\right)\right)$$
$$=\sum_{\ell\ prime}\left(\sum_{x^2<d_1d_2}F_{1,\ell}\left(\frac{d_1d_2-x^2}{4}\right)\right)\log\ell,$$

where $F_{l,\ell}\left(\frac{d_1d_2-x^2}{4}\right)$ is defined as in Theorem 1.3.

Moreover, for $-d_i$ such that $\chi_{d_i}(p)\neq -1$, so that $\mathcal{Q}_{d_i,p}$ is nonempty, one has that

$$\lim_{s\to 1}\left(\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)}\frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|}G_{p,s}(\tau_{Q_1},\tau_{Q_2})\right.$$
$$+4\pi\frac{(1+\mathrm{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)}$$
$$\left.\times\left(h_{d_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)}+h_{d_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)}-h_{d_1}h_{d_2}\phi(s)-\frac{6}{\pi}\right)\right)$$
$$=\frac{(1+\mathrm{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(2-\chi_{d_1}(p))(2-\chi_{d_2}(p))}{4}$$
$$\times\sum_{\ell\ prime}\left(\sum_{x^2<d_1d_2}F_{1,\ell}\left(\frac{d_1d_2-x^2}{4p}\right)\right)\log\ell$$
$$+\frac{(1+\mathrm{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))6h_{d_1}h_{d_2}(p-1)}{2(p+1)^2}\log p.$$

Proof. These follow from [9, Proposition 2, p. 531] specialized to $m=1$ and $N=p$, [9, Eq. (1)] and the fact that the spaces of cusp forms of weight 2 for $\Gamma_0(p)^+$ of genus zero is trivial. In particular, the reformulation of the logarithm of prime decomposition on the right hand side follows from [17, Prop. 7.12]. □

In the following two subsections, we shall use Gross et al.'s work and its extensions to establish analogous formulas for the components (2.5) and (2.6), which ultimately aid us in computing (2.4).

2.2. Component (2.5)

In this subsection, we derive a formula for the "limit" of (2.5) as s goes to 1. We start with the following lemma, which gives an interrelation between the component (2.5) and the two of those averages considered in Lemma 2.2.

Lemma 2.3. *Let the notation below be defined as before. Then for p prime, one has that*

$$\begin{aligned}
&\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\
&= (p-1-\chi_{d_1}(p)-\chi_{d_2}(p))\\
&\quad\times\left(\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{4}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}||\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} G_{1,s}(\tau_{Q_1},\tau_{Q_2})\right)\\
&\quad+\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)} \frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|} G_{p,s}(\tau_{Q_1},\tau_{Q_2}).
\end{aligned}\tag{2.8}$$

Proof. It can be easily checked that $\mathcal{Q}_{d_i}(p)/\Gamma_0(p)$ has no elliptic points for $p\geq 2$, i.e., $\Gamma_0(p)_Q=\{\pm I\}$ for any $[Q]\in\mathcal{Q}_{d_i}(p)/\Gamma_0(p)$, and thus, $|\Gamma_0(p)_Q|=2$ for $[Q]\in\mathcal{Q}_d(p)/\Gamma_0(p)$. Then using the decomposition

$$\mathcal{Q}_{d_i}=\mathcal{Q}_{d_i}(p)\sqcup\mathcal{Q}_{d_i,p},$$

one can first rewrite that

$$\begin{aligned}
&\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\
&=\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|} G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\
&=\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(p)} \frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|} G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\
&\quad-\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(p)} \frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|} G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\
&\quad-\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)} \frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|} G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\
&\quad+\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)} \frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|} G_{p,s}(\tau_{Q_1},\tau_{Q_2}),
\end{aligned}$$

and sequentially by the definition of $G_{p,s}(z_1,z_2)$ deduces that

$$\begin{aligned}
&\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\
&=\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_2}|} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2})\\
&\quad-\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_2}|} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2})\\
&\quad-\sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_2}|} \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{\gamma\in\Gamma_0(p)} g_s(\gamma\cdot\tau_{Q_1},\tau_{Q_2})\\
&\quad+\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)} \frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|} G_{p,s}(\tau_{Q_1},\tau_{Q_2})\\
&=I_1-I_2-I_3+I_4,
\end{aligned}\tag{2.9}$$

where I_i denote four of those double sums in the second-to-the-last equality, respectively.

Now note that for any $[Q_1]$, the double sum

$$\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_2}|} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2})$$

is just twice the sum of $g_s(\tau_{Q_1},z_2)$ over all imaginary quadratic points induced by $\mathcal{Q}_{d_2}$, and thus,

$$\begin{aligned}
\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_2}|} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2}) &= \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_2}|}\\
&\quad\times\sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2}).
\end{aligned}$$

Therefore, one has that

$$I_1=\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2}),\tag{2.10}$$

$$I_2=\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2}),\tag{2.11}$$

and similarly,

$$I_3 = \sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_2}|} \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\gamma\cdot\tau_{Q_1},\tau_{Q_2}). \tag{2.12}$$

Then for I_2 and I_3, by the isomorphism [10] $\mathcal{Q}_{d_i,p,\beta_i}/\Gamma_0(p) \cong \mathcal{Q}_{d_i}/\mathrm{SL}_2(\mathbb{Z})$ for some fixed $\beta_i \in \mathbb{Z}/2p\mathbb{Z}$ such that $-d_i \equiv \beta_i^2 \pmod{4p}$ if any, and the simple relation

$$\mathcal{Q}_{d_i,p} = \begin{cases} \mathcal{Q}_{d_i,p,\beta_i} \sqcup \mathcal{Q}_{d_i,p,-\beta_i} & \text{if } \chi_{d_i}(p)=1, \\ \mathcal{Q}_{d_i,p,\beta_i} & \text{if } \chi_{d_i}(p)=0, \\ \emptyset & \text{if } \chi_{d_i}(p)=-1, \end{cases}$$

one can further deduce from (2.11) and (2.12) that

$$\begin{aligned} I_2 &= (1+\chi_{d_1}(p)) \\ &\quad\times\left(\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2})\right) \\ &= (1+\chi_{d_1}(p)) \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{4}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}||\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} G_{1,s}(\tau_{Q_1},\tau_{Q_2}), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} I_3 &= (1+\chi_{d_2}(p)) \\ &\quad\times\left(\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\gamma\cdot\tau_{Q_1},\tau_{Q_2})\right) \\ &= (1+\chi_{d_2}(p)) \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{4}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}||\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} G_{1,s}(\tau_{Q_1},\tau_{Q_2}). \end{aligned} \tag{2.14}$$

So to obtain the desired identity, it remains to prove that

$$I_1 = (p+1) \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{4}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}||\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} G_{1,s}(\tau_{Q_1},\tau_{Q_2}).$$

By (2.10) it makes sense of writing the sum over $\mathcal{Q}_{d_1}/\Gamma_0(p)$ in terms of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes, that is,

$$I_1 = \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2})$$

$$= \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \left(\sum_{\substack{[Q_1']\in\mathcal{Q}_{d_1}/\Gamma_0(p) \\ [Q_1']=[Q_1] \pmod{\Gamma_0(p)}}} \frac{|\mathrm{SL}_2(\mathbb{Z})_{Q_1'}|}{|\Gamma_0(p)_{Q_1'}|} \right) \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|}$$

$$\times \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\tau_{Q_1}, \gamma\cdot\tau_{Q_2}).$$

Then it is not hard to see that $\frac{|\mathrm{SL}_2(\mathbb{Z})_{Q_1'}|}{|\Gamma_0(p)_{Q_1'}|}$ is just the ramification index of the morphism

$$\Gamma_0(p)\backslash(\mathbb{H}\cup\mathbb{Q}\cup\{i\infty\}) \to \mathrm{SL}_2(\mathbb{Z})\backslash(\mathbb{H}\cup\mathbb{Q}\cup\{i\infty\})$$

at the point $\tau_{Q_1'}$ over τ_{Q_1}, which is of degree $p+1$, and thus,

$$\sum_{\substack{[Q_1']\in\mathcal{Q}_{d_1}/\Gamma_0(p) \\ [Q_1']=[Q_1] \pmod{\Gamma_0(p)}}} \frac{|\mathrm{SL}_2(\mathbb{Z})_{Q_1'}|}{|\Gamma_0(p)_{Q_1'}|} = p+1.$$

Therefore, one finds that

$$I_1 = (p+1) \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{4}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}||\mathrm{SL}_2(\mathbb{Z})_{Q_2}|} G_{1,s}(\tau_{Q_1}, \tau_{Q_2}).$$

Finally, the desired identity (2.8) follows from the last identity and (2.9), (2.13) and (2.14). □

We are now in a position to state a formula for (2.5) that is analogous to those given in Lemma 2.2.

Lemma 2.4. *Let the notation below be defined as before. Then one has that*

$$\lim_{s\to 1} \Bigg\{ \sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)} \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} G_{p,s}(\tau_{Q_1}, \tau_{Q_2})$$

$$+ 4\pi\Bigg((p-1-\chi_{d_1}(p)-\chi_{d_2}(p))$$

$$+ \frac{(1+\mathrm{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)} \Bigg)$$

$$\times \left(h_{d_2} 2^{-s} d_1^{\frac{s}{2}} \frac{\zeta_{d_1}(s)}{\zeta(2s)} + h_{d_1} 2^{-s} d_2^{\frac{s}{2}} \frac{\zeta_{d_2}(s)}{\zeta(2s)} - h_{d_1} h_{d_2} \phi(s) - h_{d_1} h_{d_2} \frac{6}{\pi} \right) \Bigg\}$$

$$= \sum_{\ell \text{ prime}} \varepsilon_{1,\ell} \log \ell$$

$$+\frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))3h_{d_1}h_{d_2}(p-1)}{2(p+1)^2}\log p$$

where

$$\varepsilon_{1,\ell}=(p-1-\chi_{d_1}(p)-\chi_{d_2}(p))\sum_{x^2<d_1d_2}F_{1,\ell}\left(\frac{d_1d_2-x^2}{4}\right) \tag{2.15}$$
$$+C(p,d_1,d_2)\sum_{x^2<d_1d_2}F_{1,\ell}\left(\frac{d_1d_2-x^2}{4p}\right)$$

with $C(p,d_1,d_2)$ defined as in Theorem 1.3.

Proof. This follows immediately from Lemmas 2.2 and 2.3. □

2.3. Component (2.6)

The treatment of the component (2.6) is similar to that of (2.5), and we start with the following analogous interrelation.

Lemma 2.5. *Let the notation below be defined as before. Then one has that*

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},w_p\cdot\tau_{Q_2})$$
$$=\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\sum_{[Q_2]\in\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})}\frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|}G_{1,s}(\tau_{Q_1},\tau_{Q_2}) \tag{2.16}$$
$$-(1+\chi_{d_1}(p))\left(\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})}\frac{4}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}||\mathrm{SL}_2(\mathbb{Z})_{Q_2}|}G_{1,s}(\tau_{Q_1},\tau_{Q_2})\right)$$
$$+\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)}\frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|}G_{p,s}(\tau_{Q_1},\tau_{Q_2}).$$

Proof. Using the fact that for p a prime $\mathcal{Q}_{d_i}(p)/\Gamma_0(p)$ has no elliptic points, i.e., $\Gamma_0(p)_Q=\{\pm I\}$ for any $Q\in\mathcal{Q}_{d_i}$, the decomposition

$$\mathcal{Q}_{d_1}=\mathcal{Q}_{d_1}(p)\sqcup\mathcal{Q}_{d_1,p},$$

and the definition of $G_{p,s}(z_1,z_2)$ one can first rewrite that

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},w_p\cdot\tau_{Q_2})$$
$$=\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}\sum_{\gamma\in\Gamma_0(p)}g_s(\tau_{Q_1},\gamma\cdot w_p\cdot\tau_{Q_2})$$

$$= \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1}, \gamma\cdot w_p\cdot\tau_{Q_2})$$
$$- \sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1}, \gamma\cdot w_p\cdot\tau_{Q_2})$$
$$= I_1 - I_2, \tag{2.17}$$

where I_i denote two of the triple sums on the right hand side, respectively.

For I_1, note that $w_p\cdot\tau_{Q_2}$ is just a root of $Q_2\cdot\tilde{w}_p$, where $\tilde{w}_p = \sqrt{p}w_p$, so one can further rewrite the inner double sum and switch the resulting outer double sum to get

$$I_1 = \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)\cdot\tilde{w}_p/\Gamma_0(p)} \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{\gamma\in\Gamma_0(p)} g_s(\gamma\cdot\tau_{Q_1}, \tau_{Q_2}), \tag{2.18}$$

where the action of γ being shifted to the first argument in the first triple sum follows from the diagonal-$\mathrm{SL}_2(\mathbb{R})$-invariance of $g_s(z_1,z_2)$. Now by the isomorphism $\mathcal{Q}_{d_2}\cdot\tilde{w}_p/\Gamma_0(p) \cong \mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})$ given by the map

$$(aX^2+bXY+cY^2)\cdot\tilde{w}_p \to aX^2+bpXY+p^2cY^2,$$

where $\tilde{w}_p = \sqrt{p}w_p$, and the arguments used in the proof of Lemma 2.3, one can deduce from (2.18) that

$$I_1 = \sum_{[Q_2]\in\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})} \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|} \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} g_s(\gamma\cdot\tau_{Q_1}, \tau_{Q_2}). \tag{2.19}$$

For I_2, by the diagonal-$\mathrm{SL}_2(\mathbb{R})$-invariance of $g_s(z_1,z_2)$ and the fact that w_p is an involution normalizing $\Gamma_0(p)$, one can rewrite it as

$$I_2 = \sum_{[Q_1]\in\mathcal{Q}_{d_1,p}\cdot\tilde{w}_p/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1}, \gamma\cdot\tau_{Q_2}).$$

Invoking the isomorphism $\mathcal{Q}_{d_1,p}\cdot\tilde{w}_p/\Gamma_0(p) \cong \mathcal{Q}_{d_1,p}/\Gamma_0(p)$, one further obtains

$$I_2 = \sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1}, \gamma\cdot\tau_{Q_2}).$$

Then switching the order of the outer double sum in the first triple sum and using again the decomposition $\mathcal{Q}_{d_2} = \mathcal{Q}_{d_2}(p) \sqcup \mathcal{Q}_{d_2,p}$ lead to that

$$I_2 = \sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_1}|} \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(p)} \frac{2}{|\Gamma_0(p)_{Q_2}|} \sum_{\gamma\in\Gamma_0(p)} g_s(\tau_{Q_1}, \gamma\cdot\tau_{Q_2}) \tag{2.20}$$

$$-\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\frac{2}{|\Gamma_0(p)_{Q_1}|}\sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)}\frac{2}{|\Gamma_0(p)_{Q_2}|}\sum_{\gamma\in\Gamma_0(p)}g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2}).$$

Now by the isomorphism $\mathcal{Q}_{d_2}\cdot\tilde{w}_p/\Gamma_0(p)\cong\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})$ given by the map

$$(aX^2+bXY+cY^2)\cdot\tilde{w}_p\to aX^2+bpXY+p^2cY^2,$$

where $\tilde{w}_p=\sqrt{p}w_p$, and the arguments used in the proof of Lemma 2.3, one deduces from (2.20) that

$$\begin{aligned}I_2&=(1+\chi_{d_1}(p))\Bigg(\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|}\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})}\frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_2}|}\\&\quad\times\sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})}g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2})\Bigg)\\&\quad-\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\frac{2}{|\Gamma_0(p)_{Q_1}|}\sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)}\frac{2}{|\Gamma_0(p)_{Q_2}|}\sum_{\gamma\in\Gamma_0(p)}g_s(\tau_{Q_1},\gamma\cdot\tau_{Q_2})\\&=(1+\chi_{d_1}(p))\left(\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})}\frac{4}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}||\mathrm{SL}_2(\mathbb{Z})_{Q_2}|}G_{1,s}(\tau_{Q_1},\tau_{Q_2})\right)\\&\quad-\sum_{[Q_1]\in\mathcal{Q}_{d_1,p}/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2,p}/\Gamma_0(p)}\frac{4}{|\Gamma_0(p)_{Q_1}||\Gamma_0(p)_{Q_2}|}G_{p,s}(\tau_{Q_1},\tau_{Q_2}).\end{aligned}\tag{2.21}$$

Finally, the desired identity (2.16) follows from (2.17), (2.19) and (2.21). □

The following lemma is an extension of Lemma 2.2 due to Lauter and Viray [17, Cor. 1.6], and will be useful for evaluating the first component on the right hand side of (2.16).

Lemma 2.6 *(Lauter and Viray). Let the notation below be defined as before. Then for p prime, one has that*

$$\begin{aligned}&\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\sum_{[Q_2]\in\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})}\frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|}\log|j(\tau_{Q_1})-j(\tau_{Q_2})|^2\\&=\sum_{\ell\ \mathit{prime}}\nu_\ell\log\ell,\end{aligned}\tag{2.22}$$

where

$$\nu_\ell=\sum_{x^2<p^2d_1d_2}\begin{cases}F_{p,\ell}\left(\frac{p^2d_1d_2-x^2}{4}\right) & \text{if } \ell\neq p,\\ \mathfrak{A}_{p,p,2}\left(\frac{(p^2d_1d_2-x^2)}{4}\right) & \text{if } \ell=p\end{cases}$$

whose summands are defined as in Theorem 1.3.

Now recall by Lemma 2.1 that

$$\log|j(z_1)-j(z_2)|^2=\lim_{s\to 1}\left\{G_{1,s}(z_1,z_2)+4\pi\left(E_1(z_1,s)+E_2(z_2,s)-\phi(s)-\frac{6}{\pi}\right)\right\},$$

and thus, Lauter and Viary's formula (2.22) gives rise to the evaluation of

$$\lim_{s\to 1}\left\{\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\sum_{[Q_2]\in\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})}\frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|}G_{1,s}(\tau_{Q_1},\tau_{Q_2})\right.\tag{2.23}$$
$$+4\pi\left(h_{d_2}(p)\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|}E_1(\tau_{Q_1},s)+h_{d_1}\sum_{[Q_2]\in\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})}E_1(\tau_{Q_2},s)\right.$$
$$\left.\left.-h_{d_1}h_{d_2}(p)\phi(s)-h_{d_1}h_{d_2}(p)\frac{6}{\pi}\right)\right\},$$

where the first component of the limit is just the first term on the right hand side of (2.16). We next rewrite the average values of Eisenstein series $E_1(z,s)$ in the limit in terms of Dedekind zeta function and etc. The first average is given by the well known formula

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|}E_1(\tau_{Q_1},s)=2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)}.$$

The second average over $\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})$ is given as follows.

Lemma 2.7. *Let the notation below be defined as before. Then one has that*

$$\sum_{[Q]\in\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})}E_1(\tau_Q,s)=\left((1-p^{-s})(p^s-\chi_{d_2}(p))+p^{-s}(p-\chi_{d_2}(p))\right)2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)}.\tag{2.24}$$

Proof. First note that the isomorphism $\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})\cong\mathcal{Q}_{d_2}(p)/\Gamma_0(p)$ via

$$aX^2+bpXY+cp^2Y^2\to aX^2+bXY+cY^2$$

yields that

$$\sum_{\mathcal{Q}_{p^2d_2}/\mathrm{SL}_2(\mathbb{Z})}E_1(z,s)=\sum_{\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}E_1(pz,s).$$

Using the relation above and the following identity [10]

$$E_1(pz,s) = \frac{(1-p^{-2s})}{p^{-s}} E_p(z,s) + p^{-s} E_1(z,s),$$

one can deduce that

$$\begin{aligned}
&\sum_{[Q]\in \mathcal{Q}_{p^2 d_2}/\mathrm{SL}_2(\mathbb{Z})} E_1(\tau_Q, s) \\
&= \sum_{[Q]\in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_1(p\tau_Q, s) \\
&= \frac{1-p^{-2s}}{p^{-s}} \left(\sum_{[Q]\in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_p(\tau_Q, s) \right) + p^{-s} \left(\sum_{[Q]\in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_1(\tau_Q, s) \right) \\
&= \frac{1-p^{-2s}}{p^{-s}} \left(\sum_{[Q]\in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_p(\tau_Q, s) \right) + p^{-s} \left(\sum_{[Q]\in \mathcal{Q}_{d_2}/\Gamma_0(p)} E_1(\tau_Q, s) \right) \\
&\quad - p^{-s} \left(\sum_{[Q]\in \mathcal{Q}_{d_2,p}/\Gamma_0(p)} E_1(\tau_Q, s) \right) \\
&= \frac{1-p^{-2s}}{p^{-s}} \left(\sum_{[Q]\in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_p(\tau_Q, s) \right) + p^{-s}\alpha(d_2,p) \left(\sum_{[Q]\in \mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} E_1(\tau_Q, s) \right) \\
&\quad - (1+\chi_{d_2}(p))p^{-s} \left(\sum_{[Q]\in \mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} E_1(\tau_Q, s) \right) \\
&= \frac{1-p^{-2s}}{p^{-s}} \left(\sum_{[Q]\in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_p(\tau_Q, s) \right) \\
&\quad + p^{-s} \left(\alpha(d_2,p) - \chi_{d_2}(p)\right) \left(\sum_{[Q]\in \mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} E_1(\tau_Q, s) \right),
\end{aligned}$$

where $\alpha(d_2, p)$ is defined by

$$\alpha(d_2,p) = \begin{cases} \frac{p+3+2\chi_{d_2}(p)}{3} & \text{if } -d_2 = -3, \\ \frac{p+2+\chi_{d_2}(p)}{2} & \text{if } -d_2 = -4, \\ p+1 & \text{otherwise} \end{cases} \tag{2.25}$$

coming from taking elliptic points into account. The desired formula follows from the last equality above, the fact [22] that

$$(1-p^{-2s})\left(\sum_{[Q]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_p(\tau_Q,s)\right) = (1-p^{-s})(1-\chi_{d_2}(p)p^{-s})2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)}$$

and

$$\sum_{[Q]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} E_1(\tau_Q,s) = \frac{w(d_2)}{2}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)},$$

where $w(d_2)$ denotes the number of units of $\mathbb{Q}(\sqrt{-d_2})$ and the simple relation

$$(\alpha(d_2,p)-1-\chi_{d_2}(p))\frac{w(d_2)}{2} = p-\chi_{d_2}(p). \quad \square$$

Using Lemma 2.7, one can rewrite (2.23) explicitly, and consequently, obtains the following formula.

Corollary 2.8. *Let the notation below be defined as before. Then one has that*

$$\begin{aligned}
&\lim_{s\to 1}\Bigg\{\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \frac{2}{|\mathrm{SL}_2(\mathbb{Z})_{Q_1}|}G_{1,s}(\tau_{Q_1},\tau_{Q_2}) \\
&\qquad + 4\pi\Bigg(h_{d_2}(p)2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)} \\
&\qquad\qquad + h_{d_1}\left((1-p^{-s})(p^s-\chi_{d_2}(p))+p^{-s}(p-\chi_{d_2}(p))\right)2^{1-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{2\zeta(2s)} \\
&\qquad\qquad - h_{d_1}h_{d_2}(p)\phi(s) - h_{d_1}h_{d_2}(p)\frac{6}{\pi}\Bigg)\Bigg\} \\
&= \sum_{\ell\ prime}\nu_\ell\log\ell,
\end{aligned} \tag{2.26}$$

where ν_ℓ is defined as in Lemma 2.6.

Proof. This follows from Lemmas 2.1, 2.6 and 2.7. $\square$

We now come to the main result of this subsection.

Lemma 2.9. *Let the notation be defined as before. Then one has that*

$$\begin{aligned}
&\lim_{s\to 1}\Bigg\{\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} G_{p,s}(\tau_{Q_1},w_p\cdot\tau_{Q_2}) \\
&\qquad + 4\pi\Bigg(h_{d_2}(p)2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)}
\end{aligned}$$

$$
\begin{aligned}
&+ h_{d_1}\left((1-p^{-s})(p^s-\chi_{d_2}(p)) + p^{-s}(p-\chi_{d_2}(p))\right) 2^{-s} d_2^{\frac{s}{2}} \frac{\zeta_{d_2}(s)}{\zeta(2s)} \\
&\quad - h_{d_1} h_{d_2}(p)\phi(s) - h_{d_1} h_{d_2}(p)\frac{6}{\pi}\Bigg) \\
&- 4\pi\left((1+\chi_{d_1}(p)) - \frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)}\right) \\
&\quad \times \left(h_{d_2} 2^{-s} d_1^{\frac{s}{2}} \frac{\zeta_{d_1}(s)}{\zeta(2s)} + h_{d_1} 2^{-s} d_2^{\frac{s}{2}} \frac{\zeta_{d_2}(s)}{\zeta(2s)} - h_{d_1} h_{d_2}\phi(s) - h_{d_1} h_{d_2}\frac{6}{\pi}\right)\Bigg\} \\
= &\sum_{\ell \ prime} \varepsilon_{2,\ell} \log \ell \\
&+ \frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))3(p-1)h_{d_1}h_{d_2}}{2(p+1)} \log p,
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_{2,\ell} = &-(1+\chi_{d_1}(p))\left(\sum_{x^2<d_1d_2} F_{1,\ell}\left(\frac{d_1d_2-x^2}{4}\right)\right) \\
&+ \frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(2-\chi_{d_1}(p))(2-\chi_{d_2}(p))}{4} \\
&\times \left(\sum_{x^2<d_1d_2} F_{1,\ell}\left(\frac{d_1d_2-x^2}{4p}\right)\right) \\
&+ \sum_{x^2<p^2d_1d_2} \begin{cases} F_{p,\ell}\left(\frac{p^2d_1d_2-x^2}{4}\right) & \text{if } \ell \neq p, \\ \mathfrak{A}_p\left(\frac{p^2d_1d_2-x^2}{4p^2}\right) & \text{if } \ell = p \end{cases}
\end{aligned}
\tag{2.27}
$$

Proof. This follows from Lemmas 2.2 and 2.5, and Corollary 2.8. □

2.4. Component (2.7)

In this subsection, we compute the component (2.7). The treatment is similar to that given in Lemma 2.7.

Lemma 2.10. *Let the notation below be defined as before. Then one has that*

$$
\sum_{[Q]\in\mathcal{Q}_{d_i}(p)/\Gamma_0(p)} E_p^*(\tau_Q,s) = \left(p^{-s}(p-\chi_{d_i}(p)) + \frac{1-p^{-s}}{1+p^{-s}}(1-\chi_{d_i}(p)p^{-s})\right) 2^{-s} d_i^{\frac{s}{2}} \frac{\zeta_{d_i}(s)}{\zeta(2s)}. \tag{2.28}
$$

Proof. Using the relation [10]

$$E_p^*(z,s) = p^{-s}E_1(z,s) + (1-p^{-s})E_p(z,s),$$

one can show that

$$\begin{aligned}\sum_{[Q]\in\mathcal{Q}_{d_i}(p)/\Gamma_0(p)} E_1(\tau_Q,s) &= \sum_{[Q]\in\mathcal{Q}_{d_i}/\Gamma_0(p)} E_1(\tau_Q,s) - \sum_{[Q]\in\mathcal{Q}_{d_i,p}/\Gamma_0(p)} E_1(\tau_Q,s) \\ &= \alpha(d_i,p)\left(\sum_{[Q]\in\mathcal{Q}_{d_i}/\mathrm{SL}_2(\mathbb{Z})} E_1(\tau_Q,s)\right) \\ &\quad - (1+\chi_{d_i}(p))\left(\sum_{[Q]\in\mathcal{Q}_{d_i}/\mathrm{SL}_2(\mathbb{Z})} E_1(\tau_Q,s)\right) \\ &= (\alpha(d_i,p) - 1 - \chi_{d_i}(p))\frac{w(d_i)}{2}2^{-s}d_i^{\frac{s}{2}}\frac{\zeta_{d_i}(s)}{\zeta(2s)},\end{aligned}$$

where $\alpha(d_i,p)$ is defined as in (2.25). The desired formula follows from the simple relation

$$(\alpha(d_i,p) - 1 - \chi_{d_i}(p))\frac{w(d_i)}{2} = p - \chi_{d_i}(p). \quad \square$$

2.5. A limit

The verification of the following lemma is straightforward, and follows from routine computations, the famous Kronecker limit formula

$$2^{-s}d^{\frac{s}{2}}\frac{\zeta_d(s)}{\zeta(2s)} = \frac{3h_d}{\pi}\frac{1}{(s-1)} + O(1),$$

as well as the class number relation (see, e.g., [22])

$$h_{d_i}(p) = (p - \chi_{d_i}(p))h_{d_i}$$

for p prime. As such, we omit the details.

Lemma 2.11. *Let the notation be defined as before. Then one has that*

$$4\pi\lim_{s\to 1}\Bigg\{ -\Bigg((p-1-\chi_{d_1}(p)-\chi_{d_2}(p)) \\ + \frac{(1+\mathrm{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)}\Bigg)$$

$$\times \left(h_{d_2} 2^{-s} d_1^{\frac{s}{2}} \frac{\zeta_{d_1}(s)}{\zeta(2s)} + h_{d_1} 2^{-s} d_2^{\frac{s}{2}} \frac{\zeta_{d_2}(s)}{\zeta(2s)} - h_{d_1} h_{d_2} \phi(s) - h_{d_1} h_{d_2} \frac{6}{\pi} \right)$$

$$- \left(h_{d_2}(p) 2^{-s} d_1^{\frac{s}{2}} \frac{\zeta_{d_1}(s)}{\zeta(2s)} + h_{d_1} \big((1-p^{-s})(p^s - \chi_{d_2}(p)) \right.$$

$$\left. + p^{-s}(p - \chi_{d_2}(p)) \big) 2^{-s} d_2^{\frac{s}{2}} \frac{\zeta_{d_2}(s)}{\zeta(2s)} - h_{d_1} h_{d_2}(p) \phi(s) - h_{d_1} h_{d_2}(p) \frac{6}{\pi} \right)$$

$$+ \left((1+\chi_{d_1}(p)) - \frac{(1+\operatorname{sgn}(\chi_{d_1}(p) + \chi_{d_2}(p) - 1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)} \right)$$

$$\times \left(h_{d_2} 2^{-s} d_1^{\frac{s}{2}} \frac{\zeta_{d_1}(s)}{\zeta(2s)} + h_{d_1} 2^{-s} d_2^{\frac{s}{2}} \frac{\zeta_{d_2}(s)}{\zeta(2s)} - h_{d_1} h_{d_2} \phi(s) - h_{d_1} h_{d_2} \frac{6}{\pi} \right)$$

$$- \frac{1}{1-2s} \left(h_{d_2}(p) \left(p^{-s}(p - \chi_{d_1}(p)) + \frac{1-p^{-s}}{1+p^{-s}} (1 - \chi_{d_1}(p) p^{-s}) \right) 2^{-s} d_1^{\frac{s}{2}} \frac{\zeta_{d_1}(s)}{\zeta(2s)} \right.$$

$$+ h_{d_1}(p) \left(p^{-s}(p - \chi_{d_2}(p)) + \frac{1-p^{-s}}{1+p^{-s}} (1 - \chi_{d_2}(p) p^{-s}) \right) 2^{-s} d_2^{\frac{s}{2}} \frac{\zeta_{d_2}(s)}{\zeta(2s)}$$

$$\left. \left. - h_{d_1}(p) h_{d_2}(p) \left(p^{1-2s} \frac{1-p^{-1}}{1-p^{-2s}} + p^{-s} \frac{1-p^{-2s+1}}{1-p^{-2s}} \right) \phi(s) \right) \right\}$$

$$= - \frac{(1+\operatorname{sgn}(\chi_{d_1}(p) + \chi_{d_2}(p) - 1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p)) 3(p-1) h_{d_1} h_{d_2}}{(p+1)^2} \log p$$

2.6. Proof of Theorem 1.3

Relying on the preliminaries given in the preceding subsections, we are now in the position of

Proof of Theorem 1.3. We start with (2.4),

$$\sum_{[Q_1] \in \mathcal{Q}_{d_1}(p)/\Gamma_0(p)} \sum_{[Q_2] \in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \log |j_p^+(\tau_{Q_1}) - j_p^+(\tau_{Q_2})|^2$$

$$= \lim_{s \to 1} \left\{ \sum_{[Q_1] \in \mathcal{Q}_{d_1}(p)/\Gamma_0(p)} \sum_{[Q_2] \in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} G_{p,s}(\tau_{Q_1}, \tau_{Q_2}) \right.$$

$$+ \sum_{[Q_1] \in \mathcal{Q}_{d_1}(p)/\Gamma_0(p)} \sum_{[Q_2] \in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} G_{p,s}(\tau_{Q_1}, w_p \cdot \tau_{Q_2})$$

$$- \frac{4\pi}{1-2s} \left(h_{d_2}(p) \sum_{[Q_1] \in \mathcal{Q}_{d_1}(p)/\Gamma_0(p)} E_p^*(\tau_{Q_1}, s) + h_{d_1}(p) \sum_{[Q_2] \in \mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_p^*(\tau_{Q_2}, s) \right.$$

$$\left. \left. - h_{d_1}(p) h_{d_2}(p) \left(p^{1-2s} \frac{1-p^{-1}}{1-p^{-2s}} + p^{-s} \frac{1-p^{-2s+1}}{1-p^{-2s}} \right) \phi(s) \right) \right\}.$$

By Lemma 2.10, one can accordingly rewrite the sums of Eisenstein series and get

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}\log|j_p^+(\tau_{Q_1})-j_p^+(\tau_{Q_2})|^2$$

$$=\lim_{s\to 1}\Bigg\{\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},\tau_{Q_2})$$

$$+\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},w_p\cdot\tau_{Q_2})$$

$$-\frac{4\pi}{1-2s}\Bigg(h_{d_2}(p)\left(p^{-s}(p-\chi_{d_1}(p))+\frac{1-p^{-s}}{1+p^{-s}}(1-\chi_{d_1}(p)p^{-s})\right)2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)}$$

$$+h_{d_1}(p)\left(p^{-s}(p-\chi_{d_2}(p))+\frac{1-p^{-s}}{1+p^{-s}}(1-\chi_{d_2}(p)p^{-s})\right)2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)}$$

$$-h_{d_1}(p)h_{d_2}(p)\left(p^{1-2s}\frac{1-p^{-1}}{1-p^{-2s}}+p^{-s}\frac{1-p^{-2s+1}}{1-p^{-2s}}\right)\phi(s)\Bigg)\Bigg\}.$$

Then according to Lemmas 2.4, 2.9 and 2.11, one can further rewrite this as a sum of three convergent limits by some simple manipulations to get

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}\log|j_p^+(\tau_{Q_1})-j_p^+(\tau_{Q_2})|^2$$

$$=\lim_{s\to 1}\Bigg\{\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},\tau_{Q_2})$$

$$+4\pi\Bigg((p-1-\chi_{d_1}(p)-\chi_{d_2}(p))$$

$$+\frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)}\Bigg)$$

$$\times\left(h_{d_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)}+h_{d_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)}-h_{d_1}h_{d_2}\phi(s)-h_{d_1}h_{d_2}\frac{6}{\pi}\right)\Bigg\}$$

$$+\lim_{s\to 1}\Bigg\{\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)}G_{p,s}(\tau_{Q_1},w_p\cdot\tau_{Q_2})$$

$$+4\pi\Bigg(h_{d_2}(p)2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)}$$

$$+h_{d_1}\left((1-p^{-s})(p^s-\chi_{d_2}(p))+p^{-s}(p-\chi_{d_2}(p))\right)2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)}$$

$$
\begin{aligned}
&\qquad - h_{d_1}h_{d_2}(p)\phi(s) - h_{d_1}h_{d_2}(p)\frac{6}{\pi}\Big) \\
&\quad - 4\pi\left((1+\chi_{d_1}(p)) - \frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)}\right) \\
&\qquad \times \left(h_{d_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)} + h_{d_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)} - h_{d_1}h_{d_2}\phi(s) - h_{d_1}h_{d_2}\frac{6}{\pi}\right)\Bigg\} \\
&+ \lim_{s\to 1}\Bigg\{ - 4\pi\Bigg((p-1-\chi_{d_1}(p)-\chi_{d_2}(p)) \\
&+ \frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)}\Bigg) \\
&\qquad \times \left(h_{d_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)} + h_{d_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)} - h_{d_1}h_{d_2}\phi(s) - h_{d_1}h_{d_2}\frac{6}{\pi}\right) \\
&\quad - 4\pi\Bigg(h_{d_2}(p)2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)} \\
&\qquad + h_{d_1}\left((1-p^{-s})(p^s-\chi_{d_2}(p)) + p^{-s}(p-\chi_{d_2}(p))\right)2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)} \\
&\qquad - h_{d_1}h_{d_2}(p)\phi(s) - h_{d_1}h_{d_2}(p)\frac{6}{\pi}\Bigg) \\
&\quad + 4\pi\left((1+\chi_{d_1}(p)) - \frac{(1+\operatorname{sgn}(\chi_{d_1}(p)+\chi_{d_2}(p)-1))(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))}{2(p+1)}\right) \\
&\qquad \times \left(h_{d_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{d_1}(s)}{\zeta(2s)} + h_{d_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{d_2}(s)}{\zeta(2s)} - h_{d_1}h_{d_2}\phi(s) - h_{d_1}h_{d_2}\frac{6}{\pi}\right) \\
&\quad - \frac{4\pi}{1-2s}\Bigg(h_{d_2}(p)\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)} E_p^*(\tau_{Q_1},s) + h_{d_1}(p)\sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} E_p^*(\tau_{Q_2},s) \\
&\qquad - h_{d_1}(p)h_{d_2}(p)\left(p^{1-2s}\frac{1-p^{-1}}{1-p^{-2s}} + p^{-s}\frac{1-p^{-2s+1}}{1-p^{-2s}}\right)\phi(s)\Bigg)\Bigg\}.
\end{aligned}
$$

Evaluating three of these limits using Lemma 2.4, 2.9 and 2.11, one obtains that

$$
\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \log|j_p^+(\tau_{Q_1}) - j_p^+(\tau_{Q_2})|^2 = 2\sum_{\ell \text{ prime}} \mathfrak{e}_\ell \log \ell,
$$

where $\mathfrak{e}_\ell = \frac{\varepsilon_{1,\ell}+\varepsilon_{2,\ell}}{2}$ defined via (2.15) and (2.27). Dividing both sides by 2 yields the desired formula. □

2.7. Proof of Corollary 1.8

We close the current section with the proof of Corollary 1.8, which validates the first part of Conjecture 1.1.

Proof of Corollary 1.8. By Theorem 1.3 and the definition of $F_{l,\ell}(m)$, one sees that any prime factor ℓ of result $\left(H^+_{d_1,p}(x), H^+_{d_2,p}(x)\right)$ must divide the positive integers $\frac{d_1d_2-x^2}{4}$ or $\frac{p^2d_1d_2-x^2}{4}$ for some x, and thus one must have that $\ell < \frac{p^2d_1d_2}{4}$. □

3. Proof of Theorem 1.9

This section is devoted to proving Theorem 1.9, more general, Theorem 3.11. The idea of the proof is paralleled to the one used in [21, Sec. 5–8] for the case of $\Gamma_0(p)$ based on the notion of Borcherds lifts [2,3] (see, also, [15]) and Schofer's small CM value formula [19]. More specifically, by the definition (1.1) of $H^+_{d,p}(x)$ and the isomorphism $\mathcal{Q}_d(p)/\Gamma_0(p) \cong \mathrm{Cl}_d(p)$, one can first tell that

$$\log\left|\mathrm{disc}\left(H^+_{d,p}(x)\right)\right| = \sum_{\substack{[\mathfrak{c}],[\mathfrak{c}']\in\mathrm{Cl}_d(p)\\ [\mathfrak{c}]\neq[\mathfrak{c}']}} \log|j^+_p(\tau_{\mathfrak{c}}) - j^+_p(\tau_{\mathfrak{c}'})|$$
$$= -\frac{1}{4}\sum_{\substack{[\mathfrak{a}]\in\mathrm{Cl}_d(p)\\ [\mathfrak{a}]\neq[\mathcal{O}_d]}}\sum_{[\mathfrak{b}]\in\mathrm{Cl}_d(p)} -2\log|j^+_p(\tau_{\mathfrak{a}\mathfrak{b}}) - j^+_p(\tau_{\mathfrak{b}})|^2, \qquad (3.1)$$

where for an ideal $\mathfrak{a}$ with $\mathfrak{a} = \left[A, \frac{B+\sqrt{-d}}{2}\right]$, the point $\tau_{\mathfrak{a}} = \frac{B+\sqrt{-d}}{2A}$. Then if the harmonic function $-2\log|j^+_p(z_1) - j^+_p(z_2)|^2$ defined on $Y_0(p)\times Y_0(p) := \Gamma_0(p)\backslash\mathbb{H}\times\Gamma_0(p)\backslash\mathbb{H}$ can be realized as a Borcherds lift of type $(2,2)$ defined on a certain Shimura variety isomorphic to $Y_0(p)\times Y_0(p)$, and one can identify the 0-cycle

$$\sum_{[\mathfrak{b}]\in\mathrm{Cl}_d(p)}\{(\tau_{\mathfrak{a}\mathfrak{b}}, \tau_{\mathfrak{b}})\}$$

of $Y_0(p)\times Y_0(p)$ with a so-called small CM 0-cycle of this Shimura variety, one may compute the double sum (3.1) using Schofer's small CM value formula and local Whittaker functions. Thanks to Scheithauer [18], the speculation that $-2\log|j^+_p(z_1) - j^+_p(z_2)|^2$ defined on $Y_0(p)\times Y_0(p)$ is a Borcherds lift of type $(2,2)$ is actually valid. In what follows, we shall first briefly review the notion of Borcherds lifts of type $(2,2)$ and then state Scheithauer's result.

3.1. Brief review of Borcherds lifts of type $(2,2)$

Let $(V,Q) = (M_2(\mathbb{Q}), \det)$ be a rational quadratic space over $\mathbb{Q}$ of signature $(2,2)$ with general spin group

$$H = \mathrm{GSpin}(V) = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 |\ \det(g_1) = \det(g_2)\}$$

acting on V by conjugation $(g_1, g_2) \cdot v = g_1 v g_2^{-1}$. In particular, for a $\mathbb{Q}$-algebra A, write $H(A)$ for $\mathrm{GSpin}(V \otimes_{\mathbb{Q}} A)$. Denote by $\mathbb{D}$ the Grassmannian $\{[\mathbb{R}X + \mathbb{R}(-Y)] |\ \det(X) = \det(Y) < 0,\ (X, Y) = 0\}$ of oriented negative 2-planes of $V \otimes_{\mathbb{Q}} \mathbb{R}$. Then extending Q to $V \otimes_{\mathbb{Q}} \mathbb{C}$, this possesses a complex structure and can be identified with

$$\mathcal{L}/\mathbb{C}^\times = \{[Z] \in \mathbb{P}(V \otimes_{\mathbb{Q}} \mathbb{C}) |\ Q(Z) = 0,\ (Z, \overline{Z}) < 0\}$$

by $[\mathbb{R}X + \mathbb{R}(-Y)] \to [X + iY]$. Moreover, taking $\ell = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $\ell' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, one can model $\mathcal{L}/\mathbb{C}^\times$ by

$$\mathcal{K} = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & -z_2 \end{pmatrix} \in (\mathbb{Q}\ell + \mathbb{Q}\ell')^\perp \otimes_{\mathbb{Q}} \mathbb{C} |\ \mathrm{Im}(z_1)\mathrm{Im}(z_2) > 0 \right\}$$

via

$$\mathcal{K} \ni \begin{pmatrix} z_1 & 0 \\ 0 & -z_2 \end{pmatrix} \to \left[\ell' + z_1 z_2 \ell + \begin{pmatrix} z_1 & 0 \\ 0 & -z_2 \end{pmatrix} \right] = \left[\begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix} \right] \in \mathcal{L}/\mathbb{C}^\times.$$

This induces an action of H on $\mathcal{K}$ which is just the component-wise fractional linear transformation, i.e.,

$$(g_1, g_2) \cdot \begin{pmatrix} z_1 & 0 \\ 0 & -z_2 \end{pmatrix} = \begin{pmatrix} g_1 \cdot z_1 & 0 \\ 0 & -g_2 \cdot z_2 \end{pmatrix}.$$

Note that $\mathcal{K}$ has two connected components which are exactly $\mathbb{H} \times \mathbb{H}$ and $\overline{\mathbb{H}} \times \overline{\mathbb{H}}$, and we denote by $\mathcal{K}^+$ the former component.

Now for an open compact subgroup K of $H(\mathbb{A}_f)$, it is known that there is an open variety X_K of dimension 2 defined over $\mathbb{Q}$ such that

$$X_K(\mathbb{C}) \cong H(\mathbb{Q}) \backslash (\mathcal{K} \times H(\mathbb{A}_f)/K),$$

where $\mathbb{A}_f$ denotes the ring of rational finite adeles. Such a variety is called a Shimura variety. In particular, letting $\Gamma = H(\mathbb{Q})^+ \cap K$, where $H(\mathbb{Q})^+$ denotes the connected component of $H(\mathbb{Q})$ with $\det > 0$, by the strong approximation theorem, one indeed has that

$$X_K(\mathbb{C}) \cong \Gamma \backslash \mathcal{K}^+ = \Gamma \backslash (\mathbb{H} \times \mathbb{H}).$$

Example 3.1. Take

$$K = \{(g_1, g_2) \in H(\mathbb{A}_f) |\ c_1 \equiv c_2 \equiv 0 \pmod{p}\},$$

where $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, so that $H(\mathbb{Q})^+ \cap K = \Gamma_0(p) \times \Gamma_0(p)$. Then $X_K(\mathbb{C}) \cong Y_0(p) \times Y_0(p)$.

Let L be a lattice of V stabilized by K, and let L' be the dual of L. Then there is a representation ρ_L of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ called the Weil representation. For a weakly holomorphic modular form $\vec{F}(\tau) = \vec{F} : \mathbb{H} \to \mathbb{C}[L'/L]$ of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$ with respect to ρ_L invariant under K as an element in $\mathbb{C}[L'/L]$, Borcherds [2] invented a regularized theta integral lifting $\vec{F}(\tau)$ to an automorphic Green function $\Phi_L(z_1, z_2; \vec{F})$ defined in the Shimura variety $X_K(\mathbb{C})$, which is now known as a Borcherds lift of type $(2, 2)$. See also [20] for a concise summary of Borcherds' lifting theorem.

In [18], Scheithauer realizes the harmonic function $-2\log|j_p^+(z_1) - j_p^+(z_2)|^2$ as a Borcherds lift in $X_K(\mathbb{C}) \cong Y_0(p) \times Y_0(p)$ under the isomorphism given as in Example 3.1, which is now summarized in Theorem 3.2.

Theorem 3.2 *(Scheithauer). Let p be a prime such that $\Gamma_0(p)^+$ is of genus zero. Write $j_p^+(\tau) = \sum_{n=-1}^{\infty} a(n)q^n$. For $r = 0, \dots, p-1$, define $g_r(\tau)$ by*

$$g_r(\tau) = \sum_{n \equiv r \pmod p} a(n) q^{\frac{n}{p}},$$

so that

$$g_r(\tau + 1) = \zeta_p^r g_r(\tau),$$

where $\zeta_p = \exp(2\pi i/p)$, and

$$\sum_{r=0}^{p-1} g_r(\tau) = j_p^+\left(\frac{\tau}{p}\right).$$

Let $L = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, and define

$$\vec{F}(\tau) = \sum_{0 \le j,k \le p-1} F_{j,k}\phi_{\mu_{j,k}} = \sum_{0 \le j,k \le p-1} \left(\sum_{m \in \frac{jk}{p} + \mathbb{Z}} c_{j,k}(m) q^m \right) \phi_{\mu_{j,k}},$$

where $\phi_\mu = \mathrm{Char}(\mu + L)$, and

$$\left\{ \mu_{j,k} = \begin{pmatrix} 0 & j/p \\ k & 0 \end{pmatrix} + L \right\} = L'/L,$$

and for $(j, k) \neq (0, 0)$ such that $jk \equiv r \pmod p$,

$$F_{j,k}(\tau) = g_r(\tau),$$

and

$$F_{0,0}(\tau) = j_p^+(\tau) + g_0(\tau) = q^{-1} + O(q).$$

In particular, for (j, k) *such that* $jk \equiv -1 \pmod{p}$,

$$F_{j,k}(\tau) = q^{-\frac{1}{p}} + O\left(q^{\frac{p-1}{p}}\right),$$

and $c_{j,k}(0) = 0$ *for any* j, k. *Then one has that*

$$-2 \log |j_p^+(z_1) - j_p^+(z_2)|^2 = \Phi_L\left(z_1, z_2; \vec{F}\right)$$

is a Borcherds lift in $X_K(\mathbb{C}) \cong Y_0(p) \times Y_0(p)$, *where*

$$K = \{(g_1, g_2) \in H(\mathbb{A}_f) |\, c_1 \equiv c_2 \equiv 0 \pmod{p}\}.$$

As a simple consequence of Theorem 3.2, the double sum (3.1) can be rewritten as

$$\begin{aligned} &-\frac{1}{4} \sum_{\substack{[\mathfrak{a}] \in \mathrm{Cl}_d(p) \\ [\mathfrak{a}] \neq [\mathcal{O}_d]}} \sum_{[\mathfrak{b}] \in \mathrm{Cl}_d(p)} -2 \log |j_p^+(\tau_{\mathfrak{a}\mathfrak{b}}) - j_p^+(\tau_{\mathfrak{b}})|^2 \\ &= -\frac{1}{8} \sum_{\substack{[\mathfrak{a}] \in \mathrm{Cl}_d(p) \\ [\mathfrak{a}] \neq [\mathcal{O}_d]}} \sum_{(z_1, z_2) \in Z(U_{\mathfrak{a}})} \Phi_L\left(z_1, z_2; \vec{F}\right), \end{aligned}$$

where $Z(U_{\mathfrak{a}})$ is a 0-cycle of $X_K(\mathbb{C})$ identified with

$$\sum_{[\mathfrak{b}] \in \mathrm{Cl}_d(p)} \{(\tau_{\mathfrak{a}\mathfrak{b}}, \tau_{\mathfrak{b}})\} + \sum_{[\mathfrak{b}] \in \mathrm{Cl}_d(p)} \{(\tau_{\bar{\mathfrak{a}}\mathfrak{b}}, \tau_{\mathfrak{b}})\}, \tag{3.2}$$

and thus, one indeed has that

$$\log \left| \mathrm{disc}\left(H_{d,p}^+(x)\right) \right| = -\frac{1}{8} \sum_{\substack{[\mathfrak{a}] \in \mathrm{Cl}_d(p) \\ [\mathfrak{a}] \neq [\mathcal{O}_d]}} \sum_{(z_1, z_2) \in Z(U_{\mathfrak{a}})} \Phi_L\left(z_1, z_2; \vec{F}\right). \tag{3.3}$$

This boils down the problem to evaluating the average of the Borcherds lift $\Phi_L\left(z_1, z_2; \vec{F}\right)$ over $Z(U_{\mathfrak{a}})$.

3.2. $\Phi_L\left(z_1, z_2; \vec{F}\right)$ *over a small CM 0-cycle*

In his brilliant work [19, Corollary 3.5] (see, also, [4, Theorem 4.7]), Schofer establishes the following formula expressing the average value of a Borcherds lift in $X_K(\mathbb{C})$ over a special 0-dimensional subvariety $Z(U)$ of $X_K(\mathbb{C})$ called a small CM 0-cycle (see, e.g., [21, Subsection 6.1] for a definition) in terms of coefficients of Eisenstein series

$$E(\tau, s; \varphi, 1) = \sum_{m \in \mathbb{Q}} A_m(v, s, \varphi) q^m$$

for SL_2 associated to a subspace U of signature $(0,2)$ of the given rational quadratic space V (see, e.g., [21, Subsection 6.2] for a definition), which is now known as Schofer's small CM value formula and can be stated as follows (see, e.g., [21, Subsection 6.2] for definitions of unexplained notation).

Theorem 3.3 *(Schofer). Suppose that*

$$\vec{F}(\tau) = \sum_{\eta \in L'/L} \sum_{m \in -Q(\eta)+\mathbb{Z}} c(m, \eta) q^m \phi_\eta.$$

For $V = V_+ \oplus U$, where $V_+ = U^\perp$, write L_+ and L_- for $L \cap V_+$ and $L \cap U$, respectively. Write $x_\pm$ for the projections of $x \in V$ onto V_+ and U, respectively. Then

$$\begin{aligned}&\sum_{z \in Z(U)} \Phi_L(z_1, z_2; \vec{F})\\ &= \frac{4}{\mathrm{vol}(K(T))} \sum_{\lambda \in L'/(L_+ + L_-)} \sum_{m \geq 0} c(-m, \lambda + L) \sum_{\ell \in \lambda_+ + L_+} \kappa(m - Q(\ell), \lambda_- + L_-),\end{aligned}$$

where

$$\kappa(m, \varphi) = \begin{cases} \lim_{v \to \infty} A'_m(v, 0, \varphi) & \text{if } m > 0, \\ \lim_{v \to \infty} \varphi(0)\left(A'_0(v, 0, \varphi) - \log v\right) & \text{if } m = 0, \\ 0 & \text{if } m < 0, \end{cases}$$

and $A'_m(v, s, \varphi)$ is the derivative of $A_m(v, s, \varphi)$ with respect to s.

In [21, Subsection 6.3], we realize the 0-cycle $Z(U_\mathfrak{a})$ defined by (3.2) as a small CM 0-cycle of $X_K(\mathbb{C})$ in the case of Example 3.1. This together with relevant materials can be summarized as in the following lemma.

Lemma 3.4. *For $[\mathfrak{a}] \in \mathrm{Cl}_d(p) - \{[\mathcal{O}_d]\}$ with $\mathfrak{a} = \left[A, \frac{B+\sqrt{-d}}{2}\right]$ with $A > 0$ and $B^2 - 2AC = -d$ for some integer $C > 0$, let $U_\mathfrak{a}$ be the $(0,2)$-subspace associated with $[\mathfrak{a}]$ given by $U_\mathfrak{a} = \mathbb{Q} f_1^{(\mathfrak{a})} + \mathbb{Q} f_2^{(\mathfrak{a})}$, where*

$$f_1^{(\mathfrak{a})} = \begin{pmatrix} -1 & B \\ 0 & A \end{pmatrix} \quad \text{and} \quad f_2^{(\mathfrak{a})} = \begin{pmatrix} 0 & C \\ 1 & 0 \end{pmatrix}.$$

Then the 0-cycle $Z(U_\mathfrak{a})$ is a small CM 0-cycle.

In particular, in this case, the orthogonal complement $V_\mathfrak{a}$ of $U_\mathfrak{a}$ is given by $V_\mathfrak{a} = \mathbb{Q} e_1^{(\mathfrak{a})} + \mathbb{Q} e_2^{(\mathfrak{a})}$, where

$$e_1^{(\mathfrak{a})} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \quad \textit{and} \quad e_2^{(\mathfrak{a})} = \begin{pmatrix} 0 & C \\ -1 & B \end{pmatrix}.$$

Applying Schofer's theorem to the Borcherds lift given in Theorem 3.2 over the small CM 0-cycle $Z(U_{\mathfrak{a}})$ and adopting the relation (3.3), one can deduce the following formula for the discriminant of $H_{d,p}^+(x)$ in terms of $\kappa(m, \varphi)$.

Proposition 3.5. *Let the notation be defined as before. Then one has that*

$$\begin{aligned}
&\log \left| \mathrm{disc}\left(H_{d,p}^+(x) \right) \right| \\
&= -\frac{(p - \chi_d(p)) h_d}{4} \qquad (3.4)\\
&\quad \times \sum_{\substack{[\mathfrak{a}] \in \mathrm{Cl}_d(p) \\ [\mathfrak{a}] \neq [\mathcal{O}_d]}} \Bigg[\sum_{\lambda \in L/(L_+^{(\mathfrak{a})} + L_-^{(\mathfrak{a})})} \sum_{x \in \lambda_+ + L_+^{(\mathfrak{a})}} \kappa\left(1 - \det(x), \lambda_- + L_-^{(\mathfrak{a})}\right) \\
&\quad + \sum_{\substack{1 \le j,k \le p-1 \\ jk \equiv -1 \pmod{p}}} \sum_{\lambda \in L/(L_+^{(\mathfrak{a})} + L_-^{(\mathfrak{a})})} \sum_{x \in \mu_{j,k,+} + \lambda_+ + L_+^{(\mathfrak{a})}} \kappa\left(\frac{1}{p} - \det(x), \mu_{j,k,-} + \lambda_- + L_-^{(\mathfrak{a})}\right) \Bigg]
\end{aligned}$$

where $\mu_{j,k} = \begin{pmatrix} 0 & j/p \\ k & 0 \end{pmatrix}$, $L = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $L_+^{(\mathfrak{a})} = L \cap V_{\mathfrak{a}} = \mathbb{Z}e_1^{(\mathfrak{a})} + \mathbb{Z}pe_2^{(\mathfrak{a})}$, $L_-^{(\mathfrak{a})} = L \cap U_{\mathfrak{a}} = \mathbb{Z}f_1^{(\mathfrak{a})} + \mathbb{Z}pf_2^{(\mathfrak{a})}$, *and*

$$e_1^{(\mathfrak{a})} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \quad \textit{and} \quad e_2^{(\mathfrak{a})} = \begin{pmatrix} 0 & C \\ -1 & B \end{pmatrix},$$

and

$$f_1^{(\mathfrak{a})} = \begin{pmatrix} -1 & B \\ 0 & A \end{pmatrix} \quad \textit{and} \quad f_2^{(\mathfrak{a})} = \begin{pmatrix} 0 & C \\ 1 & 0 \end{pmatrix}.$$

By a careful comparison with the case of $\Gamma_0(p)$ treated in [21], one shall see that the present case differs from it by the fact that the associated vector valued modular form $\vec{F}(\tau)$ has nontrivial principal part for some components besides $F_{0,0}(\tau)$ and no nonzero constant terms while this is not the case for $\Gamma_0(p)$. This ultimately results in a slightly different set of lattice points involved.

3.3. Lattices associated with $\mathfrak{a}$

The use of (3.4) plainly relies on the knowledge of explicit expressions for the quotient lattice $L/(L_+^{(\mathfrak{a})} + L_-^{(\mathfrak{a})})$ and lattice points of cosets of $L_+^{(\mathfrak{a})}$. In [21, Subsection 7.1], we give an explicit formulation for the quotient lattice, and thus, a bit of work shall specify the lattice points for the present case. These are summarized in the following lemma.

Lemma 3.6. *Let the following notation be defined as before. Then for an ideal class* $[\mathfrak{a}] \in \mathrm{Cl}_d(p)$ *with* $\mathfrak{a} = \left[A, \frac{B+\sqrt{-d}}{2}\right]$ *chosen to be such that* $\gcd(A, pd) = 1$*, one has that*

$$\begin{aligned} &L/(L_+^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}) \\ &= \left\{ \frac{Bl}{d} e_1^{(\mathfrak{a})} - \frac{2Al}{pd}(pe_2^{(\mathfrak{a})}) + \frac{Bl}{d} f_1^{(\mathfrak{a})} - \frac{2Al}{pd}(pf_2^{(\mathfrak{a})}) + L_+^{(\mathfrak{a})} + L_-^{(\mathfrak{a})} \,\middle|\, 0 \le l \le pd-1 \right\}, \end{aligned}$$

and for

$$\lambda = \lambda_l = \frac{Bl}{d} e_1^{(\mathfrak{a})} - \frac{2Al}{pd}(pe_2^{(\mathfrak{a})}) + \frac{Bl}{d} f_1^{(\mathfrak{a})} - \frac{2Al}{pd}(pf_2^{(\mathfrak{a})}) \in L/(L_+^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}),$$

the corresponding orthogonal decomposition is given by

$$\lambda_+ = \frac{Bl}{d} e_1^{(\mathfrak{a})} - \frac{2Al}{d} e_2^{(\mathfrak{a})}, \quad \lambda_- = \frac{Bl}{d} f_1^{(\mathfrak{a})} - \frac{2Al}{d} f_2^{(\mathfrak{a})}. \tag{3.5}$$

Moreover, for $\mu_{j,k} = \begin{pmatrix} 0 & j/p \\ k & 0 \end{pmatrix}$*, one has the orthogonal decomposition*

$$\mu_{j,k,+} = \frac{B(Cpk-j)}{pd} e_1^{(\mathfrak{a})} + \frac{2A(j-Cpk)}{pd} e_2^{(\mathfrak{a})},$$

and

$$\mu_{j,k,-} = \frac{B(Cpk-j)}{pd} f_1^{(\mathfrak{a})} + \frac{2ACpk - B^2pk + 2Aj}{pd} f_2^{(\mathfrak{a})}.$$

In particular, for p *odd prime,*

$$\begin{aligned} &\mu_{j,k,-} + \lambda_{l,-} + L_-^{(\mathfrak{a})} \\ &= \frac{B(Cpk - j + pl - r_A pdk)}{pd} f_1^{(\mathfrak{a})} - \frac{2A(Cpk - j + pl - r_A pdk)}{pd} f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}, \end{aligned}$$

where $r_A = (2A)^{-1} \pmod p$*. Note that* $p \nmid (Cpk - j + pl - (2A)^{-1}pdk)$ *for* j *not divisible by* p*.*

Consequently, one obtains a more explicit form of (3.4) as follows.

Proposition 3.7. *Let the notation below be defined as before. Then for* p *odd prime, one has the following formula for the discriminant of* $H_{d,p}^+(x)$*.*

$$\begin{aligned}
&\log \left|\operatorname{disc}\left(H_{d,p}^{+}(x)\right)\right| \\
&= -\frac{(p-\chi_d(p))h_d}{4} \\
&\times \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_d(p)\\ [\mathfrak{a}]\neq[\mathcal{O}_d]}} \left[\sum_{l=0}^{pd-1} \sum_{X,Y=-\infty}^{\infty} \kappa\left(1-\frac{d(2AX+pBY)^2+(dpY-2Al)^2}{4Ad}, \frac{Bl}{d} f_1^{(\mathfrak{a})} \right.\right.\\
&\qquad\qquad \left. -\frac{2Al}{d} f_2^{(\mathfrak{a})} + L_{-}^{(\mathfrak{a})}\right) \\
&+ \sum_{\substack{1\leq j,k\leq p-1\\ jk\equiv -1 \pmod p}} \sum_{l=0}^{pd-1} \sum_{X,Y=-\infty}^{\infty} \kappa\left(\frac{1}{p} - \frac{dp^2(2AX+pBY)^2+(dp^2Y-2(pCk-j+pl)A)^2}{4Adp^2},\right. \\
&\qquad\qquad\qquad \left.\left. \frac{Bn}{pd} f_1^{(\mathfrak{a})} - \frac{2An}{pd} f_2^{(\mathfrak{a})} + L_{-}^{(\mathfrak{a})}\right)\right],
\end{aligned}$$

where $n = Cpk - j + pl - (2A)^{-1}pdk$ *with* $(2A)^{-1} \pmod p$*, and in particular,*

$$\frac{d(2AX+pBY)^2+(dpY-2Al)^2}{4A}, \frac{dp^2(2AX+pBY)^2+(dp^2Y-2(pCk-j+pl)A)^2}{4A} \in \mathbb{Z}_{\geq 0}.$$

Proof. This follows from Corollary 3.5 and the realization of lattices given as in Lemma 3.6. □

As remarked at the end of Corollary 3.4, the present case differs from the case of $\Gamma_0(p)$ by the lattice points related to components with principal part of the form $q^{-\frac{1}{p}}$. This makes the calculations of the corresponding $\kappa(m,\varphi)$ slightly more subtle as one has to consider p^2d-modulo arithmetic rather than just pd-modulo arithmetic as in the case of $\Gamma_0(p)$. In the next subsection, we accordingly refine the formulas for $\kappa(m,\varphi)$, so that they fit our needs.

Prior to closing the current subsection, several observations are noteworthy and shall facilitate us to state our final formula more concisely.

Remark 3.8. Several observations are made as follows.

(1) For l divisible by p, one must have

$$1 - \frac{d(2AX+pBY)^2+(dpY-2Al)^2}{4Ad} \neq 0,$$

otherwise, when $l=0$, $AX^2+BX(Yp)+C(Yp)^2=1$, and thus $[\mathfrak{a}]=[\mathcal{O}_d]$, a contradiction to the choice of $[\mathfrak{a}]$, or when $l=pr$ with $1\leq r\leq d-1$, one deduces that $d|4A^2p^2r^2$, and thus $d|r^2$ since $(d, 2Ap) = 1$, which implies that $d|r$ since d is square-free, a contradiction.

(2) For j not divisible by p, one must have

$$\frac{1}{p} - \frac{dp^2(2AX + pBY)^2 + (dp^2Y - 2(pCk - j + pl)A)^2}{4Adp^2} \neq 0,$$

since otherwise, one must have that $p|4j^2A^2$, a contradiction to the choices of A and j.

(3) For j not divisible by p, one can easily see that

$$\mu_{j,k,-} + \frac{Bl}{d} f_1^{(\mathfrak{a})} - \frac{2Al}{d} f_2^{(\mathfrak{a})} \not\equiv 0 \pmod{L_-^{(\mathfrak{a})}}.$$

All three of these facts together with the definition of $\kappa(m, \varphi)$ indicate that the values of

$$\kappa\left(m, \frac{Bn}{pd} f_1^{(\mathfrak{a})} - \frac{2An}{pd} f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right)$$

to be computed in Proposition 3.7 are evaluated at cases except for that of $(0, n)$ with $p \nmid n$ or $p^2|n$, and are completely given by

$$\kappa\left(m, \frac{Bn}{pd} f_1^{(\mathfrak{a})} - \frac{2An}{pd} f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right) = \lim_{v\to\infty} A'_m\left(v, 0, \frac{Bn}{pd} f_1^{(\mathfrak{a})} - \frac{2An}{pd} f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right)$$

for $0 \le n \le p^2d - 1$.

3.4. Formulas for $\kappa(m, \varphi)$

By definition, to find formulas for $\kappa\left(m, \frac{Bn}{pd} f_1^{(\mathfrak{a})} - \frac{2An}{pd} f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right)$ as n varies, it suffices to ask for formulas for $A_m\left(v, s, \frac{Bn}{pd} f_1^{(\mathfrak{a})} - \frac{2An}{pd} f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right)$. To this end, we first make the following definitions. Let

$$\Lambda(s; \chi_d) = d^{\frac{s}{2}} \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s; \chi_d)$$

be the completed Hecke L-function associated with the quadratic character χ_d known to satisfy that

$$\Lambda(1; \chi_d) = h_d,$$

where h_d is the weighted class number of $\mathbb{Q}(\sqrt{-d})$ defined as in Theorem 1.3, and let $L_p(s; \chi_d)$ be the local part of $\Lambda(s; \chi_d)$ at the finite place p defined by

$$L_p(s;\chi_d) = \frac{1}{1-\chi_d(p)p^{-s}}.$$

Also, define the local part of the divisor function of weight $-s$ associated with the quadratic character χ_d at the finite place t by

$$\sigma_{-s,t}(m,\chi_d) = \sum_{j=0}^{\mathrm{ord}_t(m)} \left(\chi_d(t)t^{-s}\right)^j,$$

and write $\Psi_{-1}(s, 4\pi mv)$ (see [16] for its precise definition) for the Archimedean component of $A_m(v,s,\varphi)$, which satisfies that $\Psi_{-1}(0,4\pi mv) = 1$. Then by [24, Theorem 5.1], formulas for

$$A_m\left(v, s, \frac{Bn}{pd}f_1^{(\mathfrak{a})} - \frac{2An}{pd}f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right)$$

can be stated as follows.

Theorem 3.9. *For a prime t, write $|N|_t$ for the t-adic norm of integer N. Define $W_{\mathfrak{a}}(s,m,n)$ by*

(1) for $p^2|n$,

$$W_{\mathfrak{a}}(s,m,n) = \begin{cases} 1+\left(\frac{-Am}{p}\right)p^{-s} & \text{if } \mathrm{ord}_p(m)=0, \\ 1+(p-1)\left(\sum_{j=2}^{\mathrm{ord}_p(m)} \left(\chi_d(p)p^{-s}\right)^j\right) - \left(\chi_d(p)p^{-s}\right)^{\mathrm{ord}_p(m)+1} & \text{if } \mathrm{ord}_p(m)\geq 1, \\ 1+(p-1)\sum_{n=2}^{\infty}\chi_d(p)^n p^{-ns} & \text{otherwise,} \end{cases}$$

(2) for $p|n$ and $p^2\nmid n$,

$$W_{\mathfrak{a}}(s,m,n) = 1+\left(\frac{-Ad\left(md+A(n/p)^2\right)}{p}\right)p^{-s},$$

(3) for $p\nmid n$,

$$W_{\mathfrak{a}}(s,m,n) = 1.$$

Then for $n \in \mathbb{Z}/p^2d\mathbb{Z}$, one has that for $m > 0$ such that $m \in -\frac{An^2}{p^2d} + \mathbb{Z}$,

$$\Lambda(s+1;\chi_d)A_m\left(v, s, \frac{Bn}{pd}f_1^{(\mathfrak{a})} - \frac{2An}{pd}f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right)$$
$$= -2\prod_{t\nmid pd}\sigma_{-s,t}(m,\chi_d)\prod_{\substack{t|d\\ \mathrm{ord}_t(md)>0}}(1+(-d,-mN(\mathfrak{a}))_t|md|_t^s)L_p(s+1;\chi_d)\frac{1}{p}W_{\mathfrak{a}}(s,m,n)$$
$$\times (2m\sqrt{d\pi v})^s\Psi_{-1}(s,4\pi mv),$$

and for $m = 0$,

$$A_0\left(v, s, \frac{Bn}{pd}f_1^{(\mathfrak{a})} - \frac{2An}{pd}f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right)$$
$$= v^{\frac{s}{2}}\delta_{0,n} - \sqrt{\pi}v^{-\frac{s}{2}}d^{-\frac{1}{2}}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}\frac{L(s;\chi_d)}{L(s+1;\chi_d)}\frac{L_p(s+1;\chi_d)}{L_p(s;\chi_d)}\frac{1}{p}W_{\mathfrak{a}}(s,0,n),$$

where $\delta_{0,n}$ is the Kronecker delta indicator.

As a consequence of Theorem 3.9, one can derive the following "prime-factorization" formulas for

$$\kappa\left(m, \frac{Bn}{pd}f_1^{(\mathfrak{a})} - \frac{2An}{pd}f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right).$$

It is worth first noting that as indicated in Remark 3.8, for $m = 0$, one only has to consider the number n's for which $p|n$ and $p^2 \nmid n$.

Corollary 3.10. *Let the following notation be defined as before. Then one has that*

(1) for $m > 0$ such that $m \in -\frac{An^2}{p^2d} + \mathbb{Z}$,

$$\kappa\left(m, \frac{Bn}{pd}f_1^{(\mathfrak{a})} - \frac{2An}{pd}f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right) = \frac{2}{h_d}\sum_{\ell\ prime}\mathcal{G}_{\ell,\mathfrak{a},n}(m)\log\ell,$$

where

(a) for $\ell \nmid pd$,

$$\mathcal{G}_{\ell,\mathfrak{a},n}(m) = \prod_{t\nmid \ell pd}\sigma_{0,t}(m,\chi_d)\prod_{\substack{t|d\\ \mathrm{ord}_t(md)>0}}(1+(-d,-mN(\mathfrak{a}))_t)$$
$$\times\frac{1}{p-\chi_d(p)}W_{\mathfrak{a}}(0,m,n)\sum_{j=1}^{\mathrm{ord}_\ell(m)}\chi_d(\ell)^j j,$$

(b) for $\ell | d$,

$$\mathcal{G}_{\ell,\mathfrak{a},n}(m) = \prod_{t \nmid pd} \sigma_{0,t}(m,\chi_d) \prod_{\substack{t|d/\ell \\ \mathrm{ord}_t(md)>0}} (1 + (-d, -mN(\mathfrak{a}))_t) \times \frac{1}{p-\chi_d(p)} W_{\mathfrak{a}}(0,m,n)(-d,-m)_\ell \mathrm{ord}_\ell(md),$$

(c) and for $\ell = p$,

$$\mathcal{G}_{\ell,\mathfrak{a},n}(m) = \prod_{t \nmid pd} \sigma_{0,t}(m,\chi_d) \times \prod_{\substack{t|d \\ \mathrm{ord}_t(md)>0}} (1 + (-d, -mN(\mathfrak{a}))_t) \frac{1}{p-\chi_d(p)} \frac{W'_{\mathfrak{a}}(0,m,n)}{\log p},$$

(2) and for $m = 0$ *and* n *such that* $p|n$ *and* $p^2 \nmid n$,

$$\kappa\left(0, \frac{Bn}{pd} f_1^{(\mathfrak{a})} - \frac{2An}{pd} f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right) = -\frac{2}{p-\chi_d(p)} \log p.$$

Proof. These follow directly from the definition of $\kappa(m,\varphi)$ and Theorem 3.9. In particular, for $m = 0$ and n such that $p|n$ and $p^2 \nmid n$,

$$W_{\mathfrak{a}}(s,0,n) = 1 + \left(\frac{-dA^2(n/p)^2}{p}\right) p^{-s} = 1 + \chi_d(p)p^{-s},$$

and thus,

$$\begin{aligned} &A_0\left(v, s, \frac{Bn}{pd} f_1^{(\mathfrak{a})} - \frac{2An}{pd} f_2^{(\mathfrak{a})} + L_-^{(\mathfrak{a})}\right) \\ &= -\sqrt{\pi} v^{-\frac{s}{2}} d^{-\frac{1}{2}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \frac{L(s;\chi_d)}{L(s+1;\chi_d)} \frac{L_p(s+1;\chi_d)}{L_p(s;\chi_d)} \frac{1}{p} \left(1 + \chi_d(p)p^{-s}\right) \\ &= -\sqrt{\pi} v^{-\frac{s}{2}} d^{-\frac{1}{2}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \frac{L(s;\chi_d)}{L(s+1;\chi_d)} L_p(s+1;\chi_d) \frac{1}{p} \left(1 - p^{-2s}\right). \quad \square \end{aligned}$$

We are now ready to state and prove the general formula for the prime factorization of the discriminant of $H_{d,p}^+(x)$.

Theorem 3.11. *Let* $p \geq 3$ *be a prime such that* $\Gamma_0(p)^+$ *is of genus zero, and let* $-d$ *be an odd fundamental discriminant coprime to* p. *Then*

$$\log\left|\mathrm{disc}\left(H_{d,p}^+(x)\right)\right| = \sum_{\ell \text{ prime}} \mathfrak{c}_\ell \log \ell,$$

where

$$\mathfrak{e}_\ell = \sum_{\substack{[\mathfrak{a}] \in \mathrm{Cl}_d(p) \\ [\mathfrak{a}] \neq [\mathcal{O}_d]}} \mathfrak{e}_{\ell,\mathfrak{a}},$$

and

$$\begin{aligned} \mathfrak{e}_{\ell,\mathfrak{a}} = &\sum_{l=0}^{pd-1} \sum_{X,Y=-\infty}^{\infty} G_{\ell,\mathfrak{a},pl}\left(\frac{4Ad - d(2AX+pBY)^2 - (dpY - 2Al)^2}{4Ad}\right) \\ &+ \sum_{\substack{1 \le j,k \le p-1 \\ jk \equiv -1 \pmod p}} \sum_{l=0}^{pd-1} \\ &\times \sum_{X,Y=-\infty}^{\infty} G_{\ell,\mathfrak{a},j}\left(\frac{4Adp - dp^2(2AX+pBY)^2 - (dp^2Y - 2(pCk - j + pl)A)^2}{4Adp^2}\right), \end{aligned}$$

the ideal class representatives $\mathfrak{a}$ *are chosen to be* $\left[A, \frac{B+\sqrt{-d}}{2}\right]$ *so that* $\gcd(A, pd) = 1$*, and the function* $G_{\ell,\mathfrak{a},n}(m)$ *is defined as follows (all the products involved are over primes prescribed by the corresponding given assumption).*

- *(i) For* $m < 0$, $G_{\ell,\mathfrak{a},n}(m) = 0$*. Similarly, as noted in Remark 1.10, this implies* $\mathfrak{e}_{\ell,\mathfrak{a}}$*'s are all finite sums.*
- *(ii) For* $m = 0$ *and* n *such that* $p|n$ *and* $p^2 \nmid n$*,*
  - *(a) if* $\ell \neq p$, $G_{\ell,\mathfrak{a},n}(0) = 0$*,*
  - *(b) otherwise,* $G_{\ell,\mathfrak{a},n}(0) = \frac{h_d}{2}$*.*
- *(iii) For* $m > 0$ *and*
  - *(a) for* $\ell \nmid pd$*,*

$$G_{\ell,\mathfrak{a},n}(m) = -\frac{1}{2} \prod_{t \nmid \ell pd} \left(\sum_{j=0}^{\mathrm{ord}_t(m)} \chi_d(t)^j \right) \times \prod_{\substack{t|d \\ \mathrm{ord}_t(md)>0}} \left(1 + (-d, -mN(\mathfrak{a}))_t\right) W_{\mathfrak{a}}(0,m,n) \sum_{j=1}^{\mathrm{ord}_\ell(m)} \chi_d(\ell)^j j,$$

(b) for $\ell | d$,

$$G_{\ell,\mathfrak{a},n}(m) = -\frac{1}{2}\prod_{t\nmid pd}\left(\sum_{j=0}^{\mathrm{ord}_t(m)}\chi_d(t)^j\right) \times \prod_{\substack{t|d/\ell \\ \mathrm{ord}_t(md)>0}}(1+(-d,-mN(\mathfrak{a}))_t)\, W_{\mathfrak{a}}(0,m,n)(-d,-m)_\ell \mathrm{ord}_\ell(md),$$

(c) for $\ell = p$,

$$G_{\ell,\mathfrak{a},n}(m) = -\frac{1}{2}\prod_{t\nmid pd}\left(\sum_{j=0}^{\mathrm{ord}_t(m)}\chi_d(t)^j\right) \times \prod_{\substack{t|d \\ \mathrm{ord}_t(md)>0}}(1+(-d,-mN(\mathfrak{a}))_t)\,\frac{W'_{\mathfrak{a}}(0,m,n)}{\log p},$$

where $W_{\mathfrak{a}}(s,m,n)$ *is defined as in Theorem 3.9.*

Proof. This follows from Proposition 3.7 and Corollary 3.10. □

Theorem 3.11 implies Corollary 3.12 and as a result, affirms Conjecture 1.1 (2).

Corollary 3.12. *Any prime factor of the discriminant of* $H^+_{d,p}(x)$ *with* p *an odd prime and* d *coprime to* $2p$ *is less than* dp.

Proof. By Theorem 3.11, especially the coefficients $G_{\ell,\mathfrak{a},n}(m)$ defined therein, for a prime ℓ dividing disc $\left(H^+_{d,p}(x)\right)$, one can see that either $\ell|d$, $\ell = p$,

$$\ell \left| \left(d - \frac{d(2AX+pBY)^2 + (dpY - 2Al)^2}{4A}\right) \le d,\right.$$

or

$$\ell \left| \left(dp - \frac{dp^2(2AX+pBY)^2 + (dp^2Y - 2(pCk - j + pl)A)^2}{4A}\right) \le dp.\right.$$

Therefore, any prime factor of disc $\left(H^+_{d,p}(x)\right)$ with p odd prime and $(d,2p) = 1$ is bounded by dp. □

Remark 3.13. In fact, Corollary 3.12 can directly follow from Corollary 3.5, which holds for all underlying primes p and any negative fundamental discriminant $-d$, Lemma 3.6, as well as local properties of the function $\kappa(m,\varphi)$. In addition, by a careful inspection of the argument used in [21, Section 7.1], one can see that the first assertion of Lemma 3.6 indeed holds for arbitrary prime p and negative fundamental discriminant $-d$ with A chosen such that $\gcd(A, pd) = 1$, and as an implication, one can further deduce that the values of $\kappa(m, \varphi)$ in (3.4) of Corollary 3.5 are all given by evaluations at nonnegative integer m of the form $\frac{N}{pd}$ with $N \leq pd$. Furthermore, by Corollary 3.5 and the definition of $\kappa(m, \varphi)$, one can tell that a prime ℓ divides $\left|\mathrm{disc}\left(H^{+}_{d,p}(x)\right)\right|$ only if the ℓ-part of $\kappa(m, \varphi)$ vanishes or $\ell | pd$, and thus by [16, Section 4], only if $\mathrm{ord}_{\ell}\left(\frac{N-r}{pd}\right) > 0$ for some $r < N$, which indicates that $0 < N - r \leq pd$, or $\ell | pd$. Therefore, one can ultimately conclude that any prime factor of $\left|\mathrm{disc}\left(H^{+}_{d,p}(x)\right)\right|$ is bounded by pd, and hence, Conjecture 1.1 (2) does hold for arbitrary cases.

Data availability

Data will be made available on request.

Appendix A. Magma codes for Theorems 1.3 and 3.11

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Magma code for Theorem 1.3

```
p:=7;
d1:=3;
d2:=4;
Z:=IntegerRing();
L:=[];

for x in [0..d1*d2] do
num:=(d1*d2-x^2)/(4);
if num in Z and num gt 0 then
for l in [1..num] do
if IsPrime(l) and num/l in Z then
Include(~L,l);
end if;
end for;
end if;
end for;

for x in [0..p^2*d1*d2] do
num:=(p^2*d1*d2-x^2)/(4);
if num in Z and num gt 0 then
for l in [1..num] do
if IsPrime(l) and num/l in Z then
Include(~L,l);
end if;
```

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```
end for;
end if;
end for;

for x in [0..d1*d2] do
num:=(d1*d2-x^2)/(4*p);
if num in Z and num gt 0 then
for l in [1..num] do
if IsPrime(l) and num/l in Z then
Include(~L,l);
end if;
end for;
end if;
end for;

if p notin L then
Include(~L,p);
end if;

L:=Sort(L);

//epsilon
epsilon:= function(N,ell)
if N in Z and (d1/ell) in Z then
e:=2;
elif N notin Z then
 e:= 0;
elif (d1*p/ell) notin Z and Valuation(N,ell) mod 2 eq 1 then
 e:= 0;
else e:= 1;
end if;
return e;
end function;

// A
A:=function(m,l,r,ell)

SetQ:=[];
for q in [1..m] do
  if IsPrime(q) and m/q in Z and q ne ell then
    Include(~SetQ,q);
  end if;
end for;

prod:=1;
for q in SetQ do
if q eq 2 then
a:=0;
elif LegendreSymbol(-d1,q) eq 1 and q ne l then
    a:=1+Valuation(m,q);
elif LegendreSymbol(-d1,q) eq 1 and q eq l then
    a:=2;
elif LegendreSymbol(-d1,q) eq -1 and q ne l then
    a:= 1/2*(1+(-1)^Valuation(m,q));
elif d1/q in Z and HilbertSymbol(-d1, -Numerator(m), q) eq 1 and q ne l
 then
    a:=2;
elif d1/q in Z and HilbertSymbol(-d1, -Numerator(m), q) eq 1 and q eq l
```

```
and Valuation(m,q) le 2 then
     a:=1;
else a:=0;
  end if;
prod*:= a;
end for;

return epsilon(m/(ell^r),ell)*prod;

end function;

//sgn
sgn:= function(r)
if r ge 0 then
s:=1;
else s:=-1;
end if;
return s;
end function;

//F
F:=function(l,ell,m)
K:=QuadraticField(-d1);
D:=Decomposition(K,ell)[1][1];
e:=RamificationDegree(D);
prod:=0;
for r in [1..Valuation(m,ell)] do
prod+:=A(m,l,r,ell);
end for;

return 1/e*prod;
end function;

//e
el:=function(ell)
sumF1:=0;

for x in [-d1*d2..d1*d2] do
  if x^2 lt d1*d2 and ((d1*d2-x^2)/4) in Z then
    sumF1+:= F(1,ell,(d1*d2-x^2)/4);
  end if;
end for;

alpha:=(p-2-2*LegendreSymbol(-d1,p)-LegendreSymbol(-d2,p))/2*sumF1;


sumF2:=0;

for x in [-d1*d2..d1*d2] do
  if x^2 lt d1*d2 and ((d1*d2-x^2)/(4*p)) in Z then

    sumF2+:= F(1,ell,(d1*d2-x^2)/(4*p));

  end if;
end for;

beta:= (1+sgn(LegendreSymbol(-d1,p)+LegendreSymbol(-d2,p)-1))
*(1+LegendreSymbol(-d1,p))*(1+LegendreSymbol(-d2,p))
```

```
*(2-LegendreSymbol(-d1,p))*(2-LegendreSymbol(-d2,p))/4*sumF2;

sumF3:=0;

for x in [-p^2*d1*d2..p^2*d1*d2] do
if x^2 lt p^2*d1*d2 and ((p^2*d1*d2-x^2)/(4)) in Z then
if ell ne p then
sumF3+:= F(p,ell,(p^2*d1*d2-x^2)/4);
elif ell eq p and (p^2*d1*d2-x^2)/(4*p^2) in Z then
 sumF3+:= A((p^2*d1*d2-x^2)/4,p,2,p);
end if;
end if;
end for;

gamma:=1/2*sumF3;

return alpha+beta+gamma;

end function;

//return the result bla
bla:=0;
for ell in L do
  if el(ell) ne 0 then
  printf "%o^%o *", ell, el(ell);
  end if;
end for;
```

Magma code for Theorem 3.11

```
p:=7;
d:=3;

A:=31;
B:=11;
C:=1;
Z:=IntegerRing();

Y1u:=Floor((Sqrt(4*A*d)+2*A*(p*d-1))/(d*p));
Y1l:=Ceiling(-Sqrt(4*A*d)/(d*p));
X1u:=Floor((Sqrt(4*A)-B*p*Y1l)/(2*A));
X1l:=Ceiling((-Sqrt(4*A)-B*p*Y1u)/(2*A));

Y2u:=Floor((Sqrt(4*A*d*p)+2*(p*C*(p-1)-1+p*(p*d-1))*A)/(d*p*p));
Y2l:=Ceiling((-Sqrt(4*A*d*p)+2*(p*C-(p-1))*A)/(d*p*p));
X2u:=Floor((Sqrt(4*A/p)-p*B*Y2l)/(d*p*p));
X2l:=Ceiling((-Sqrt(4*A/p)-p*B*Y2u)/(2*A));

ell:=3;

K:=QuadraticField(-d);

CLdp:=RingClassGroup(K,p);

if d eq 3 then
h:=1/3;
elif d eq 4 then
h:=1/2;
else h:=ClassNumber(K);
```

```
end if;

//list of ell
L:=[];
for ell in [1..d*p] do
 if IsPrime(ell) then
  Include(~L, ell);
  end if;
end for;
 L:=Sort(L);

//W_{p,a}(s,m,n)
W:=function(p,s,m,n)
if n/p notin Z then
  w:=1;
  elif n/p in Z and n/p^2 notin Z then
    w:= 1+KroneckerSymbol(-Numerator(A*d*(m*d+A*(n/p)^2))*
    Denominator(A*d*(m*d+A*(n/p)^2)),p)*p^(-s);
  elif n/p^2 in Z then
     if Valuation(m,p) eq 0 then
        w:=1+KroneckerSymbol(-Numerator(A*m)*Denominator(A*m),p)*p^(-s);
        elif Valuation(m,p) eq 1 then
        w:=1-(KroneckerSymbol(-d,p)*(p^(-s)))^(Valuation(m,p)+1);
        elif Valuation(m,p) ge 2 then
          sum:=0;
         for j in [2..Valuation(m,p)] do
            sum+:=((KroneckerSymbol(-d,p)*(p^(-s)))^j);
         end for;
       w:= 1+(p-1)*sum-(KroneckerSymbol(-d,p)*(p^(-s)))^(Valuation(m,p)+1);
 else w:= 1+(p-1)*
 ((KroneckerSymbol(-d,p)*(p^(-s)))^2/(1-(KroneckerSymbol(-d,p)*(p^(-s)))));

       end if;

end if;
return w;
end function;

//W'_{p,a}(s,m,n)
Wd:=function(p,s,m,n)
if n/p notin Z then
  wd:=0;
  elif n/p in Z and n/p^2 notin Z then
    wd:= -KroneckerSymbol(-Numerator(A*d*(m*d+A*(n/p)^2))*
    Denominator(A*d*(m*d+A*(n/p)^2)),p)*p^(-s)*Log(p);
  elif n/p^2 in Z then
     if Valuation(m,p) eq 0 then
        wd:=-KroneckerSymbol(-Numerator(A*m)*Denominator(A*m),p)*p^(-s)*
        Log(p);
        elif Valuation(m,p) eq 1 then
        wd:=(KroneckerSymbol(-d,p)^(Valuation(m,p)+1)*(Valuation(m,p)+1)*
        p^(-(Valuation(m,p)+1)*s)*Log(p));
        elif Valuation(m,p) ge 2 then
          sum:=0;
         for j in [2..Valuation(m,p)] do
            sum+:= KroneckerSymbol(-d,p)^j*j*p^(-s*j)*Log(p);
         end for;
        wd:= -(p-1)*sum +(KroneckerSymbol(-d,p)^(Valuation(m,p)+1)*
```

```
      (Valuation(m,p)+1)*p^(-(Valuation(m,p)+1)*s)*Log(p));
      end if;
     else wd:= -(p-1)*(-KroneckerSymbol(-d,p)^2*2*p^(-2*s)*Log(p)*
     (1-KroneckerSymbol(-d,p)*p^(-s))-(KroneckerSymbol(-d,p)*p^(-s))^2*
     KroneckerSymbol(-d,p)*p^(-s)*Log(p))/(1-KroneckerSymbol(-d,p)*
     p^(-s))^2;
     printf "error";
end if;
return wd;
end function;

//sigma_{s,q}(m, chi)
sigma:=function(s,q,m)
sig:=0;
for j in [0..Valuation(m,q)] do
sig+:=(KroneckerSymbol(-d,q)*q^s)^j;
end for;
return sig;
end function;

//GLANM(ell,a,n,m);
GLANM:=function(ell,n,m)
Sigma:=1;
prod:=1;
if p*d/ell notin Z then
  for q in [1..ell*p*d] do
    if ell*p*d/q notin Z and IsPrime(q) then
    Sigma*:=sigma(0,q,m);
    end if;
  end for;
  for q in [1..d] do
    if d/q in Z and IsPrime(q) then
    if Valuation(m*d,q) gt 0 then
    prod*:=(1+HilbertSymbol(-d/1, -m/1, q));
    end if;
    end if;
  end for;
  sum:=0;
  for j in [1..Valuation(m, ell)] do
  sum+:=KroneckerSymbol(-d, ell)^j*j;
  end for;
  prod:=prod*W(p,0,m,n)*sum;
elif d/ell in Z then
 for q in [1..p*d] do
 if p*d/q notin Z and IsPrime(q) then
 Sigma*:=sigma(0,q,m);
 end if;
 end for;
 for q in [1..d/ell] do
  if IsPrime(q) then
  if d/(ell*q) in Z and Valuation(m*d,q) gt 0 then
    prod*:=(1+HilbertSymbol(-d/1,-m/1,q));
  end if;
  end if;
 end for;
prod:= prod*W(p,0,m,n)*HilbertSymbol(-d/1,-m/1,ell)*Valuation(m*d,ell);
elif ell eq p then
  for q in [1..p*d] do
```

```
    if p*d/q notin Z and IsPrime(q) then
      Sigma*:=sigma(0,q,m);
    end if;
  end for;
for q in [1..d] do
 if d/q in Z and IsPrime(q) then
 if Valuation(m*d,q) gt 0 then
  prod*:= (1+HilbertSymbol(-d/1,-m/1,q));
 end if;
end if;

end for;
prod:=prod*Wd(p,0,m,n)/Log(p);

end if;

g:=-1/2*Sigma*prod;

return g;
end function;

//G_{ell,a,n}(m)
G:=function(ell,n,m)
 g:=0;
if m gt 0 then
  g:=GLANM(ell,n,m);
elif m eq 0 and n/p in Z and n/p^2 notin Z then
  if ell ne p then
    g:=0;
  elif ell eq p then
    g:=h/2;
  end if;
elif m lt 0 then
  g:=0;
end if;
return g;
end function;

power:=function(ell)
sum1:=0;
sum2:=0;

for l in [0..p*d-1] do
 for y in [Y1l..Y1u] do
for x in [X1l..X1u] do
 if (4*A*d-d*(2*A*x+B*y*p)^2-(d*y*p-2*A*l)^2)/(4*A*d) ge 0 then
m:=(4*A*d-d*(2*A*x+B*y*p)^2-(d*y*p-2*A*l)^2)/(4*A*d);
sum1+:=G(ell,p*l,m);
 end if;
 end for;
 end for;
end for;

 for j,k in [1..p-1] do
  if j*k mod p eq p-1 then
  for l in [0..p*d-1] do
   for y in [Y2l..Y2u] do
for x in [X2l..X2u] do
```

```
   if (4*A*d*p - d*p^2*(2*A*x+p*B*y)^2-(d*p^2*y-2*(p*C*k-j+p*l)*A)^2) ge 0
   then
  m:= (4*A*d*p - d*p^2*(2*A*x+p*B*y)^2
       -(d*p^2*y-2*(p*C*k-j+p*l)*A)^2)/(4*A*d*p^2);
  sum2+:=G(ell,j,m);
 end if;
end for;
   end for;
   end for;
  end if;
  end for;

 return Floor(sum1+sum2);
end function;

for ell in L do
if power(ell) ne 0 then
printf "%o^%o *", ell, power(ell);
end if;
end for;
```

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